LAGRANGIAN FIBRATIONS BY PRYM VARIETIES

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A dissertation submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the College of Arts and Sciences.

Chapel Hill
2020

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ABSTRACT

Chen Shen: Lagrangian Fibrations by Prym Varieties
(Under the direction of Justin Sawon)

We describe and study Lagrangian fibrations including non-compact ones from Hitchin systems and compact ones from the moduli spaces of semi-stable sheaves of some fixed invariant on K3 surfaces. We prove properties of the relative Prym variety $\mathcal{P}$ constructed from a double cover of a degree one del Pezzo surface T by a K3 surface S using the bi-anticanonical class of T. $\mathcal{P}$ is a 6 dimensional singular holomorphic symplectic variety with generic fibers being Prym varieties of a new polarization type $(1, 2, 2)$. We examine the singularities of $\mathcal{P}$ and show that $\mathcal{P}$ is birational to a quotient of a smooth simply-connected projective variety by an involution. In the end, we prove a degeneration from the relative Prym variety associated to del Pezzo surfaces to a natural compactification of the $\mathrm{Sp}(2n, \mathbb{C})$-Hitchin system.

To my parents.

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor Justin Sawon for suggesting this problem and the valuable discussions and guidance.

On this project, the communication with Prakash Belkale, Jiuzu Hong and Shrawn Kumar were insightful. Also, special thanks to Gulia Sacca for her interest and clarification in email correspondence.

In earlier years, it was through the classes, seminars, workshops and numerous online resources that I explored mathematics. Having wonderful colleagues and caring friends has made this journey more cheerful. It was fun to be with Claire Kiers and Josh Kiers, Marc Besson and lots of others. At Chapel hill, the free and peaceful environment nourishes me and will remain in my mind forever.

Along this way, the best thing is to always have the love and understanding from my parents. I wish to express my endless gratitude to their constant support.

TABLE OF CONTENTS

CHAPTER 1

Introduction

The object of study in this article is Lagrangian fibrations from different sources, both compact and non-compact, including Jacobian fibrations and Prym fibrations as fibration structures of some holomorphic symplectic varieties such that fibers are maximal isotropic spaces of the symplectic form.

The classical Hitchin system and its extension to other gauge groups provide various examples of non-compact Lagrangian fibrations. They stemmed from Hitchin's solution to the self-duality equation on a Riemann surface in the differential geometry category, and were later transformed to the moduli space of stable bundles under the algebro-geometric point of view. The moduli space of Higgs bundles admits hyperkähler structure, and the Hitchin system is a Lagrangian fibration which maps the moduli space onto a complex space with half the dimension. Generic fibers of $\mathrm{GL}(n, \mathbb{C})$-Hitchin system are Jacobian varieties, and of other gauge groups are more general abelian varieties, namely Prym varieties.

The examples we will illustrate on the compact side are Lagrangian fibrations of a special class of varieties, the irreducible holomorphic symplectic varieties, smooth and non-smooth.

A compact Kähler manifold X is irreducible holomorphic symplectic if there is a unique holomorphic 2-form up to scaling which is symplectic and X is simply-connected. The existence of the symplectic form imposes the dimension of X to be even. In dimension 2, they are $K3$ surfaces. In higher dimensions, only two series of examples are known, the Hilbert scheme of n points on $K3$ surfaces $S^{[n]}$ and the generalized Kummer variety A_n, together with two sporadic examples, O'Grady's 6 and 10 dimensional resolution space of some moduli space of stable sheaves on an abelian surface and a K3 surface.

Irreducible holomorphic symplectic variety is a fundamental class of compact Kähler manifold with trivial first Chern class.

Theorem 1.0.1. *(Beauville - Bogomolov decomposition theorem) Y is a compact Kähler manifold with $c_1(Y) = 0$, then there exists an étale finite cover*

$$\pi : A \times \prod_i CY_i \times \prod_j X_j \to Y$$

where A is a complex torus, each CY_i is a Calabi-Yau manifold of dimension greater than 2 and each X_j is an irreducible holomorphic symplectic manifold.

A surprising fact about irreducible holomorphic symplectic manifolds is shown in a result of Matsushita, which says the only fibration they admit are Lagrangian fibrations.

Theorem 1.0.2. *(Matsushita) X is a $2n$ dimensional irreducible holomorphic symplectic manifold with symplectic form σ. For any proper morphism $f\colon X \to B$ with connected fibers of positive dimension, the following is satisfied*

1. *f is equi-dimensional, $\dim B = n$,*
2. *Generic fibers of the map are n dimensional abelian varieties,*
3. *Any fiber is a Lagrangian subvariety and thus f is Lagrangian.*

Theorem 1.0.3. *(Hwang) In the context of the thoerem of Matsushita, if B is smooth, then $B = \mathbb{P}^n$.*

We focus on the moduli space of semi-stable sheaves $\mathcal{M}_{v,H}(S)$ on $K3$ surface S, which is holomorphic symplectic due to a remarkable result of Mukai and is deformation equivalent to $S^{[n]}$.

In search of a wider class of examples, D. Markushevich and A.S. Tikhomirov proposed a way to construct Lagrangian fibration by Prym varieties as an extension of such systems [MT]. It is a subspace of the moduli space of sheaves on the K3 surface, called the relative Prym variety.

Later, the approach was systematized by E. Arbarello, A. Ferretti and G. Sacca in [AFS]. The key ingredients are a K3 surface S with an anti-symplectic involution τ, which induces a double cover $\pi : S \to T$, together with a linear system $\pi^*|C'|$ on the K3 surface which is invariant under the involution τ. The relative Prym variety $\mathcal{P}_{v,C}(S)$ is defined to be the fixed locus of some symplectic involution on $\mathcal{M}_{v,C}(S)$, where C is a smooth curve in $\pi^*|C'|$ and $v = (0, [C], 1 - g(C))$.

By a classification of Nikulin, the quotient of the K3 surface under a anti-symplectic involution could be Enriques surface, del Pezzo surfaces and so on. The example in [AFS] studied the Enriques surface case. D. Markushevich and A.S. Tikhomirov in [MT] and G. Menet in [MEN] looked into the degree 2 del Pezzo surface case, and degree 3 del Pezzo surface case was investigated by T. Matteini in [MAT]. They are all singular irreducible holomorphic symplectic varieties, except the hyperelliptic case on the Enriques surface, which admits a symplectic resolution of type $S^{[n]}$.

Following this framework, we examined the properties of the relative Prym variety $\mathcal{P}_{v,C}(S)$ associated to a degree 1 del Pezzo surface and the linear system $\pi^*|-2K_T|$. This gives a Lagrangian fibration in Prym varieties of a new polarization type $(1,2,2)$, which is dual to the generic fibers in Matteini's system. We leave it as a conjecture that the new system is dual to a degeneration of some Matteini system. Interestingly, in this case, the Mukai vector $v = (0, [C], 1 - g(C))$ is non-primitive and the linear system $|C|$ on S is hyperelliptic.

Theorem 1.0.4. *Let $\pi : S \to T, \Delta_T$ be a double cover from a K3 surface S to a degree 1 del Pezzo surface T ramified along a smooth curve Δ_T. Let C be a smooth curve in $\pi^*|-2K_T|$, $g(C) = 5$.*

Denote by $\mathcal{P}_{v,C}(S)$ the relative Prym variety associated to π and $v = (0, [C], 1 - g(C))$.

$\mathcal{P}_{v,C}(S)$ is a 6 dimensional holomorphic symplectic variety with symplectic singularities.

The Lagrangian fibration is given by the support map $\mathcal{P}_{v,C}(S) \to \pi^*|-2K_T|$. Its generic fibers are Prym varieties of polarization type $(1,2,2)$, each of which is a Prym variety of a double cover of curves $\pi|_{C_0} : C_0 \to C_0'$, where $C_0 \in |C|$ with genus 5 and $C_0' \in |-2K_T|$ with genus 2.

The singularity of $\mathcal{P}_{v,C}(S)$ consists of 120 points of type $\mathbb{C}^6 / \pm 1$, a $\mathbb{P}^2$ family of type $\mathbb{C}^4 \times (\mathbb{C}^2 / \pm 1)$, and a quadric cone Λ family generically of type $\mathbb{C}^4 \times (\mathbb{C}^2 / \pm 1)$.

Lastly, $\mathcal{P}_{v,C}(S)$ is birational to a quotient of a smooth projective variety by a regular involution $\tilde{\mathcal{M}}/\tilde{\iota}$. $\tilde{\mathcal{M}}$ is a smooth projective variety birational to a smooth irreducible holomorphic symplectic variety which is a Beauville-Mukai system fibered by Jacobians.

Using this birational model, we can show $h^{2,0}(\tilde{\mathcal{P}}) = 1$ for any desingularization $\tilde{\mathcal{P}}$ of $\mathcal{P}$, and under the assumption that $\mathcal{P}$ only has quotient singularities, $\pi_1(\mathcal{P}) = 0$ or $\mathbb{Z}_2$.

Lastly, we explore relations between these two types of Lagrangian fibrations. For the $\mathrm{GL}(n,\mathbb{C})$-Hitchin system and Beauville-Mukai system, Ron Donagi, Lawrence Ein, Robert Lazarsfeld described a degeneration from the latter to a compactification of the former in some cases in [DEL]. We extend this relation to some Lagrangian fibrations of Prym varieties, namely, from some $\mathrm{Sp}(2n,\mathbb{C})$-Hitchin system to some relative Prym variety associated to del Pezzo surfaces.

Theorem 1.0.5. *Let $\pi : S \to T, \Delta_T$ be a double cover from a K3 surface S to a del Pezzo surface T of degree d ramified over a smooth curve Δ_T. Let $C = \pi^{-1}(\Delta_T)$, then $g(C) = g = d + 1$.*

Denote by $\mathcal{H}$ the $\mathrm{Sp}(2n)$-Hitchin system of stable pairs on C consisting of a rank $2n$, degree d vector *bundle and a Higgs field ϕ, the Hitchin map is a Lagragian fibration*

$$\mathcal{H} \to \mathbb{C}^{\tilde{g}}$$

where $\tilde{g} = n(2n+1)g$.

And denote by $\mathcal{P}$ the relative Prym variety $\mathcal{P}_{v,C}(S)$ constructed from $\pi : S \to T, \Delta_T$ and a smooth curve in $\pi^*|-2nK_T| \subset |2nC|$, using $v = (0, [2nC], k+1-g(2nC))$, where $k = d + ((2n)^2 - 2n)(g-1)$. The support map is a Lagrangian fibration

$$\mathcal{P} \to \mathbb{P}^{\tilde{g}}$$

Then there is a degeneration from $\mathcal{P}$ to a natural compactification of $\mathcal{H}$ given by a support map of schemes over $\mathbb{P}^1$

$$\bar{\mathcal{W}} \to \bar{\mathcal{B}}$$

which satisfies for $t \neq 0 \in \mathbb{P}^1$, the fiber

$$[\bar{\mathcal{W}}_t \to \bar{\mathcal{B}}_t] \cong [\mathcal{P} \to \mathbb{P}^{\tilde{g}}]$$

and

$$[\bar{\mathcal{W}}_0 \to \bar{\mathcal{B}}_0] \cong [\bar{\mathcal{H}} \to \mathbb{P}^{\tilde{g}}].$$

The paper is composed of three parts.

Chapter 2 is the first part devoted to non-compact Lagrangian fibrations from Hitchin systems. We give description of the $\mathrm{GL}(n,\mathbb{C})$- Hitchin system, followed by the extension to other gauge groups. We introduce the notion of spectral data, which gives local information of the fiber. Later the spectral data will become a bridge to relate the non-compact and compact systems.

From Chapter 3 to Chapter 7, we discuss compact Lagrangian fibrations. Chapter 3 is some review of irreducible holomorphic symplectic varieties and their properties, including a subsection on general facts about K3 surfaces. From Chapter 4, we merge into the main object in this paper, moduli space of stable pure sheaves of dimension one on K3 surface. Section 5 states the systematized construction of the relative Prym variety that alters the Jacobian fibration into a Prym fibration. We revisit the result of the relative Prym associated to Enriques surface [AFS], degree two del Pezzo surface [MT, MEN] and degree three del Pezzo surface [MAT].

We present our original work on degree one del Pezzo surface in Chapter 6. Due to the non-primitive linear system we choose, the geometry between the surfaces are more involved. The singularities of the relative Prym variety occur in the non-integral elements in the linear system we pick on the K3 surface. Finally, we build a birational model for the relative Prym variety, which is a quotient of a smooth simply-connected space birational to an irreducible holomorphic symplectic variety by an involution. We show $h^{2,0}$ of any resolution of the relative Prym variety is 1. Additionally, section 7 is aimed to explain the motivation in considering such space, namely, we suspect this relative Prym variety associated to degree one del Pezzo surface is "dual" to the relative Prym of some special degree three del Pezzo surface. We utilize the Pantazis's construction to analyze the duality for each curve double cover. It still remains open whether the duality fits into a family from del Pezzo surface.

Lastly in Chapter 8, we are interested in answering the question of how to relate the compact and non-compact Lagrangian fibrations. Inspired by the original work in [DEL], which gives a degeneration from the Beauville-Mukai system to a compactification of $\mathrm{GL}(n,\mathbb{C})$-Hitchin system, we managed to extend the degeneration to Prym fibrations, namely, from the relative Prym variety associated to del Pezzo surface to $\mathrm{Sp}(2n,\mathbb{C})$-Hitchin system.

CHAPTER 2

Hitchin systems

There are many circumstances where non-compact Lagrangian fibrations arise, including lots of examples from representation theory. Another main class of examples is the Hitchin systems.

Hitchin system was first discovered by Nigel Hitchin in 1987 when he studied the solutions to the Yang-Mills self-duality equations on a Riemann surface C. The Hitchin moduli space $\mathcal{H}$ consisting of stable Hitchin pairs is a partial compactification of the cotangent bundle $\mathrm{T}^* \mathcal{U}_{n,d}$ of the moduli space of stable bundles on C as a complex curve. [H87]

2.1 Moduli space of stable vector bundles on a curve

We let the genus g of C to be ≥ 1, fix integers n and d, assuming $(n, d) = 1$ and denote by

$$\mathcal{U}_{n,d}(C)$$

the moduli space of rank n semi-stable vector bundles of degree d on C.

A vector bundle E is *semi-stable* if every sub-bundle $F \subset E$ satisfies

$$\frac{degF}{rkF} \leq \frac{degE}{rkE}$$

and *stable* if strict inequality holds. The ratio is called the slope of the bundle

$$\mu(E) = \frac{degE}{rkE}$$

The construction of this moduli space involves using geometric invariant theory (GIT). In general, the space could be delicate, but when $(n, d) = 1$ the notions of semi-stability and stability agree, so $\mathcal{U}_{n,d}(C)$ is a fine moduli space. In particular, it is smooth and projective.

Applying Riemann-Roch theorem to the tangent space of $\mathcal{U}_{n,d}(C)$ at E, which is $H^1(C, \mathrm{End}E)$, together with $H^0(C, \mathrm{End}E) = \mathbb{C}$, the moduli space has dimension

$$\dim \mathcal{U}_{n,d}(C) = n^2(g-1) + 1$$

2.2 $\mathrm{GL}(n, \mathbb{C})$-Hitchin system

The Hitchin system is related to the cotangent bundle

$$\mathrm{T}^* \mathcal{U} := \mathrm{T}^* \mathcal{U}_{n,d}(C)$$

of the moduli space discussed in the first section.

A *Higgs pair* or a *Higgs bundle* is (E, ϕ), where E is a holomorphic vector bundle of rank n and degree d on C, and $\phi \in H^0(C, \mathrm{End}(E) \otimes K_C)$, which is called a *Higgs field*.

By deformation theory, if E is a stable vector bundle in $\mathcal{U}$

$$\mathrm{T}_E \mathcal{U} = H^1(C, \mathrm{End}(E))$$

and Serre duality shows

$$H^1(C, \mathrm{End}(E)) = H^0(C, \mathrm{End}(E) \otimes K_C)^\vee$$

so ϕ corresponds to a direction in the cotangent space at E.

The Hitchin map is taking the coefficients of the characteristic polynomial of the Higgs field

$$h : \mathrm{T}^* \mathcal{U} \to \mathcal{B} = \bigoplus_{i=1}^{n} H^0(C, K_C^i)$$

$$(E, \phi) \mapsto (-\operatorname{tr}\phi, \operatorname{tr} \wedge^2 \phi, \ldots, (-1)^n \det \phi)$$

In order to projectivize $h : \mathrm{T}^* \mathcal{U} \to \mathcal{B}$, we need stability conditions on Higgs pairs or Higgs bundles.

The Higgs bundle (E, ϕ) is *(semi-)stable* if for any ϕ-invariant sub-bundle $F \subset E$, i.e. $\phi(F) \subset F \otimes K_C$

$$\mu(F) < (=) \mu(E)$$

It turns out the moduli space of semi-stable Higgs bundles $\mathcal{H}$ is a smooth quasi-projective variety, with $T^*\mathcal{U}$ being an open subvariety [H87]. If E is a stable vector bundle, then it is automatic that (E, ϕ) is a stable Higgs bundle.

The moduli space $\mathcal{H}$ possesses a holomorphic symplectic structure σ on the infinitesimal deformations $(\dot{E}, \dot{\phi})$ of a Higgs bundle (E, ϕ)

$$\sigma((\dot{E}_1, \dot{\phi}_1), (\dot{E}_2, \dot{\phi}_2)) = \int_C tr(\dot{E}_1\dot{\phi}_2 - \dot{\phi}_1\dot{E}_2)$$

where $\dot{E} \in \Omega^{0,1}(C, \mathrm{End}E)$ and $\dot{\phi} \in \Omega^{1,0}(C, \mathrm{End}E)$[LS].

The extension

$$h : \mathcal{H} \to \mathcal{B}$$

is the Hitchin system. The Hitchin map h is a Lagrangian fibration. Furthermore, it is a proper algebraically completely integrable system.

Note,

$$\frac{1}{2}\dim\mathcal{H} = n^2(g-1) + 1$$

and

$$\dim\mathcal{B} = n^2(g-1) + 1$$

by taking alternating sum of Riemann-Roch formula, the fibration is equidimensional.

2.3 Spectral data

In order to understand the geometry of the fibration h, we first introduce spectral curves, which are embedded in the total space of the canonical bundle K_C and are in one-one correspondence with points in the Hitchin base.

Take $p = (p_1, \ldots, p_n) \in B$, let $\pi : K_C \to C$ be the projection and consider the tautological section x of the pull back π^*K_C. The zeroes of

$$x^n + p_1x^{n-1} + \cdots + p_n = 0$$

are n sections of the canonical bundle K_C, as eigenvalues of ϕ in the fiber of h above p. Define the union of them to be the spectral curve D_p. It is mapping n to one onto C. The characteristic polynomial of ϕ restricted to an invariant sub-bundle would divide the characteristic polynomial.

2.3.1 Generic fibration

The fibers of the Hitchin map could be very complicated depending on the smoothness of the spectral curve. Generically, $p \in \mathcal{B}$ corresponds to a smooth spectral curve D_p and the fiber of the Hitchin map can be recognized as

$$\mathrm{Jac}^k(D_p)$$

the class of degree k line bundles on a smooth irreducible curve, where $k = d + (n^2 - n)(g - 1)$ [H87].

The dimension of $\mathrm{Jac}^k(D_p)$ is equal to the genus of D_p, that is $n^2(g-1)+1$ due to the adjunction formula. Denote the total space of the canonical bundle $tot(K_C)$ by X,

$$K_X \cdot D_p + D_p^2 = 2g(D_p) - 2$$

$K_X = 0$ as X is symplectic, D_p is in the linear system $|nC| \in tot(K_C)$ and $K_C^2 = 2g - 2$, hence

$$2g(D_p) - 2 = 2n^2(g - 1)$$

Under the push forward of $\pi : D_p \to C$, $\pi_*(L)$ for $L \in \mathrm{Jac}^k(D_p)$ recovers the rank n stable vector bundle E on C of degree d. It is automatically stable because generically, the equation for the spectral curve is irreducible, thus there is no ϕ variant sub-bundle. One can use the Grothendick-Riemann-Roch formula to check $k = d + (n^2 - n)(g - 1)$.

The correspondence

$$(D_p, L) \longleftrightarrow (C, E, \phi).$$

is discussed in detail in [H07, BNR].

2.4 Other $\mathrm{G}_{\mathbb{C}}$-Hitchin systems

Vector bundles corresponds to principal $\mathrm{GL}(n, \mathbb{C})$-bundles. For other simple Lie group $G_{\mathbb{C}}$, the semi-stability of principal $G_{\mathbb{C}}$-bundle P is equivalent to the semi-stability of the vector bundle ad P associated to the adjoint representation.

For example when $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$, the moduli space of semi-stable $\mathrm{SL}(n, \mathbb{C})$-bundles is the fiber of the determinant morphism $\det : \mathcal{U}_{n,d}(C) \to \mathrm{Jac}^d(C)$

$$\mathcal{U}_{n,d}(C)^{\mathrm{SL}} = \det^{-1}(\Lambda)$$

in fact, it is independent of the element $\Lambda \in \mathrm{Jac}(C)$.

In the $G_{\mathbb{C}}$-Hitchin moduli space, a $\mathrm{G}_{\mathbb{C}}$-*Higgs bundle* (P, ϕ) consists of a principal $G_{\mathbb{C}}$-bundle P and $\phi \in H^0(C, \mathrm{ad}P \otimes K_C)$, where $\mathrm{ad}P$ is a bundle of Lie algebra $\mathfrak{g}$. The stability for $\mathrm{G}_{\mathbb{C}}$-Higgs bundle can be defined, but generic Higgs bundles do not have any ϕ-invariant sub-bundle, so there is no destabilizing sub-bundle and we refer the details to [LS]. There exists moduli space of semi-stable $G_{\mathbb{C}}$-Higgs bundles $\mathcal{H}_{G_{\mathbb{C}}}$.

The Hitchin map on $\mathcal{H}_{G_{\mathbb{C}}}$ is defined using the basis for the ring of invariant polynomials $p_i, i = 1, \ldots, k$ on the Lie algebra $\mathfrak{g}$

$$h : \mathcal{H}_{G_{\mathbb{C}}} \to \mathcal{B}_{G_{\mathbb{C}}} = \bigoplus_{i=1}^{k} H^0(C, K_C^{d_i})$$

$$(E, \phi) \mapsto (p_1(\phi), \ldots, p_k(\phi))$$

The Higgs field reflects the specialty of the automorphism group and the dimension of $\mathcal{H}_{G_{\mathbb{C}}}$ can be calculated by

$$\dim \mathcal{H}_{G_{\mathbb{C}}} = 2 \dim H^0(C, \mathrm{ad}P \otimes K_C) = 2(\dim G_{\mathbb{C}})(g-1)$$

The Lagrangian fibration structure of the Hitchin moduli space is similar, except that the generic fiber will be more general abelian varieties.

$\mathbf{G} = \mathrm{SL}(\mathbf{n}, \mathbb{C})$

A $\mathrm{SL}(n, \mathbb{C})$ - Higgs bundle is a classical Higgs bundle (E, ϕ) such that E is a stable rank n vector bundle whose determinant bundle is trivial and Higgs field ϕ has trace zero, so the Hitchin base is

$$\mathcal{B} = \bigoplus_{i=2}^{n} H^0(C, K_C^i)$$

Notice

$$\begin{aligned}\dim \mathcal{B} &= \bigoplus_{i=1}^{n} H^0(C, K_C^i) - \dim H^0(C, K_C)\\ &= n^2(g-1) + 1 - g\\ &= (n^2-1)(g-1)\end{aligned}$$

The generic fiber of the Hitchin map is the kernel of the norm map $\mathrm{Jac}(D_p) \to \mathrm{Jac}(C)$ induced by the n to one cover $\pi : D_p \to C$, which is a Prym variety of dimension

$$\dim \mathrm{Jac}(D_p) - \dim \mathrm{Jac}(C) = n^2(g-1) + 1 - g = (n^2-1)(g-1).$$

See details in [H07].

$\mathbf{G} = \mathrm{Sp}(\mathbf{2n}, \mathbb{C})$

A $\mathrm{Sp}(2n, \mathbb{C})$-Higgs bundle is a stable vector bundle E of rank $2n$ with a symplectic form $\langle , \rangle$ and a holomorphic section $\phi \in H^0(C, \mathrm{End}E \otimes K_C)$ such that $\langle \phi v, w \rangle = -\langle v, \phi w \rangle$, for $v, w \in E$. As eigenvalue of $\mathfrak{sp}(2n, \mathbb{C})$ comes in $\pm\lambda$ pairs, the characteristic polynomial of ϕ has the form

$$x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n}$$

So the Hitchin map H is fibered over

$$B = \bigoplus_{i=1}^{n} H^0(C, K_C^{2i})$$

For generic $p \in B$, the spectral curve D_p is described by the equation $x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n} = 0$ which is irreducible. Note that it is invariant under an involution $\sigma(x) = -x$ on K_C, which also induces an involution on the Jacobian of the smooth curve D_p. So $h^{-1}(p)$ is the subset of line bundles $L \in \mathrm{Jac}^d(D_p)$ such that $\sigma^* L \cong L^*$, that is, the Prym variety of the map $D_p \to D_p/\sigma$. Putting this into a diagram, we have

$$\begin{tikzcd} D_p \arrow[r, hook] \arrow[d, "2:1"'] & tot(K_C) \arrow[d, "2:1"] \\ D_p/\sigma \arrow[r, hook] & tot(K_C^2) \end{tikzcd}$$

Calculating the genera of the curves, we have the dimension of the fiber Prym varieties equal to $n(2n+1)(g-1)$, which is equal to $\dim \mathcal{B}$. Details can be found in a later section.

$\mathbf{G} = \mathrm{SO}(\mathbf{2n+1}, \mathbb{C})$

A $\mathrm{SO}(2n+1, \mathbb{C})$-Hitchin pair consists of a stable vector bundle E of rank $2n+1$ with a non-degenerate symmetric bilinear form $(,)$ and a holomorphic section $\phi \in H^0(C, \mathrm{End}(E) \otimes K_C)$ which satisfies $(\phi v, w) = -(v, \phi w)$. Its characteristic polynomial has the form

$$x(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n})$$

where $a_2, \ldots, a_{2m}$ are derived from invariant polynomials on Lie algebra of $\mathrm{SO}(2n+1)$, $a_{2i} \in H^0(C, K_C^{2i})$.

A vector bundle homomorphism ϕ always has an eigenvalue zero, so we obtain a line bundle $E_0 \subset E$ as the eigenspace. And $0 \to E_0 \to E \to E_1$, then if we take out E_1, it is a rank $2n$ bundle with a skew form $\langle v, w \rangle = (\phi v, w)$, thus sending us back to the $Sp(2n)$ case. The Hitchin base space has the same dimension $n(2n+1)(g-1)$.

Sp($2n$) and SO($2n+1$) Hitchin systems have dual Lagrangian fibrations, i.e. the generic fibers are dual abelian varieties, using the Langlands duality for Sp($2n$) and SO($2n+1$) [H07].

The long-term goal of this project is to find compact analogy for these $G_{\mathbb{C}}$-Hitchin systems and the dualities between those compact analogies.

CHAPTER 3

Irreducible holomorphic symplectic varieties

3.1 Irreducible holomorphic symplectic varieties

The compact Lagrangian fibrations in this paper are from irreducible holomorphic symplectic varieties.

In dimension one, complex curves can be classified into three classes by the curvature. Gauss-Bonnet Theorem reveals that they correspond to genus zero curves, elliptic curves and higher genus curves. Likewise in higher dimension, we are interested in compact Kähler manifolds with vanishing first Chern class.

The following theorem tells us the building blocks of this class up to a finite cover.

Theorem 3.1.1. *(Beauville - Bogomolov decomposition theorem) Y is a compact Kähler manifold with* $c_1(Y) = 0$*, then there exists an unramified finite cover*

$$\pi : A \times \prod_i CY_i \times \prod_j X_j \to Y$$

where A is complex tori, each CY_i *is Calabi-Yau manifold of dimension greater than 2 and each* X_j *is irreducible holomorphic symplectic manifold.*

It follows from Yau's proof of the Calabi conjecture. Using the holonomy group of the unique Ricci-flat metric in the Kähler class, one has the classification into three types.

A Calabi-Yau manifold is a complex Kähler manifold with a nowhere vanishing holomorphic n-form, where n is the dimension of the manifold. K3 surfaces and complex torus of dimension 2 are Calabi-Yau manifolds in dimension 2.

Definition 3.1.2. *A compact Kähler manifold X is holomorphic symplectic if it has a unique holomorphic 2-form up to scalar which is a symplectic form, i.e. closed and non-degenerate*

$$H^0(X, \Omega_X^2) = \mathbb{C}\sigma$$

Furthermore, if X is simply-connected, X is irreducible holomorphic symplectic.

Remark 3.1.3. *We can define it on smooth complex compact varieties which are meromorphic image of compact Kähler manifolds, and call it (irreducible) holomorphic symplectic varieties.*

By definition, a holomorphic symplectic manifold X is always of even dimension $2n$. The non-degeneracy of σ means σ^n is a nowhere vanishing section of the canonical bundle K_X, and thus K_X is trivial. In particular, $c_1(X) = 0$.

If X is holomorphic symplectic but not irreducible holomorphic symplectic, then by the Beauville - Bogomolov decomposition theorem, it may have an irreducible piece in the decomposition of its finite cover.

K3 surfaces and complex torus of dimension 2 are holomorphic symplectic manifolds in dimension 2, but only K3 surfaces are 2-dimensional irreducible holomorphic symplectic manifolds as complex tori are not simply-connected.

Example 3.1.4. *In higher dimension, the first sequence of examples of irreducible holomorphic symplectic manifolds are constructed out of a projective K3 surface S by Beauville, which is the Hilbert scheme $S^{[n]}$ of length n sub-schemes. [Bea]*

It is natural to consider direct products of lower dimension examples, so consider $S^n = S \times S \times \cdots \times S$, and

$$cp_1^*\sigma_S + cp_2^*\sigma_S + \cdots + cp_n^*\sigma_S$$

where σ_S is the 2-form on S and $c \in \mathbb{C}$ is a holomorphic 2-form. Thus, we take a quotient

$$Sym^n(S) = S^n/\mathfrak{S}_n$$

and it turns out there is no other holomorphic 2-forms on $Sym^n(S)$. Then take the desingularization of it along diagonal and achieve the Hilbert scheme of n points on S.

The symplectic form on $S^{[n]}$ is deduced from S, see [OGHK].

Example 3.1.5. *Another series of examples is constructed out of an abelian surface A, called the generalized Kummer variety $K_n(A), n \geq 2$. It is the pre-image of 0 under the summation map*

$$s_{n+1} : A^{[n+1]} \to A$$

For $n = 1$ this is the Kummer surface of A, a K3 surface.

These two series are not deformation equivalent as $b_2(S^{[n]}) = 23$ and $b_2(K_n(A)) = 7$.

Theorem 3.1.6. *(Huybrechts) If two irreducible holomorphic symplectic manifolds X_1, X_2 are birational, then they are deformation equivalent, thus diffeomorphic.*

The moduli space of stable sheaves on K3 surface when smooth and compact is irreducible holomorphic symplectic. In fact, they are deformation equivalent to $S^{[n]}$, thus called of $S^{[n]}$-type [MUK, Y99].

There are only two other known examples of holomorphic symplectic manifolds not deformation equivalent to $S^{[n]}$ and $K_n(A)$. O'Grady found the two other examples by a desingularization of some moduli space of semi-stable sheaves on K3 surface and abelian surface, respectively OG(6) and OG(10), as they are 6 and 10 dimensional moduli spaces.

Remark 3.1.7. *There are four known types of smooth irreducible holomorphic symplectic manifolds up to deformation equivalence: $S^{[n]}$, $K_n(A)$, $OG(6)$, $OG(10)$.*

Mathematicians involved in the study of irreducible holomorphic manifolds include Bogomolov, Fujiki, Beauville, Verbitsky and Voisin. Lots of their work are from algebro-geometric point of view, such as global Torelli Theorem, Chow ring and so on, while our main interest is the Lagrangian fibration structure of some of these examples, in particular, the moduli space of sheaves on K3 surfaces.

3.2 K3 surfaces

K3 surface can be defined for arbitrary field, but we will restrict to complex numbers $\mathbb{C}$. It was rediscovered by A. Weil in 1958 and named after Kummer, Kähler and Kodaira.

Definition 3.2.1. *A K3 surface is a compact smooth surface S such that the canonical bundle $K_S = \Omega_S^2 \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$.*

They represent a Ricci flat class in the classification of algebraic surfaces. The others are the positively curved del Pezzo surfaces and the negatively curved surfaces of general type. Del Pezzo surfaces have models and are already classified, while surfaces of general types are far from being completely described.

Let ω on S be a holomorphic section of K_S without zero locus. ω is closed and thus serves as a symplectic form on S. Also ω is an isomorphism between $\Omega_S \simeq \mathcal{T}_S$.

Also, although not directly required in its definition, S has to be a Kähler manifold. The proof is due to Yum-Tong Siu. So K3 surfaces are Calabi Yau manifolds.

K3 surfaces are all diffeomorphic. We look into examples of projective K3 surfaces.

Example 3.2.2. *(Fermat quartic) The Fermat quartic surface F is cut out by a quartic polynomial in $\mathbb{P}^3$,*

$$x^4 + y^4 + z^4 + w^4 = 0.$$

It is smooth and by adjunction formula, one can show the canonical class on F is trivial, $K_F = K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(F) = \mathcal{O}_{\mathbb{P}^3}(-4+4)|_F = \mathcal{O}_F$. To show $H^1(F, \mathcal{O}_F) = 0$, we use the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_F \to 0,$$

and Kodaira vanishing $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$.

For a K3 surface S with polarization L, define the *genus of S* by $L^2 = 2g - 2$. Using the restriction of the natural polarization of $\mathbb{P}^3$, F is a K3 surface with degree 4 and genus 3.

As Fermat quartic is simply connected, all K3 surfaces are simply-connected, so K3 surfaces are the most simple irreducible holomorphic symplectic manifold.

Example 3.2.3. *For similar reason, when $\sum d_i = n + 3$, smooth complete intersection of hypersurfaces of degree $d_1, \ldots, d_n$ in $\mathbb{P}^{n+2}$ is a K3 surface. When $n = 2$, $n_1 = 2, n_2 = 3$ gives a K3 surface with degree 6 genus 4 . When $n = 3$, $n_1 = n_2 = n_3$ gives a K3 surface of degree 8.*

Example 3.2.4. *(Double plane) Consider a double cover of the projective plane $\mathbb{P}^2$ branched along a smooth sextic*

$$\pi : S \to \mathbb{P}^2, s_6$$

The canonical bundle formula shows

$$K_S \simeq \pi^*(K_{\mathbb{P}^2} \otimes \mathcal{O}(3)) \simeq \mathcal{O}_S.$$

Also

$$\pi_* \mathcal{O}_S \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$$

indicating $H^1(S, \mathcal{O}_S) = 0$.

Using $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ as the polarization, S is K3 of degree 2 and genus 2.

Example 3.2.5. *(Kummer surface) Let A be an abelian surface and consider the involution $i : x \mapsto -x$ on A with 16 fixed points. The quotient A/i of this involution has 16 rational double point singularities, but admits a resolution by S by blowing up. In fact, S is a K3 surface.*

To prove this, consider the diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, "bl"] \arrow[d, "\pi"] & A \arrow[d] \\ S \arrow[r, "bl"] & A/i \end{tikzcd}$$

where $\tilde{A}$ is the blow up of A at the 16 fixed points, and π is a double cover along the exceptional divisors E_i.

Using the canonical bundle formula for blow up, $\omega_{\tilde{A}} \simeq O_A(\sum E_i)$. Let $\bar{E}_i$ be the image of E_i under π, then $\pi^*(\bar{E}_i) = 2E_i$ and $\omega_{\tilde{A}} \simeq \pi^*\omega_S \otimes O_A(\sum E_i)$, which together show $\pi^*\omega_S \simeq O_{\tilde{A}}$.

π is determined by a square root L of $O(\sum \bar{E}_i)$, then using $\pi_* O_{\tilde{A}} \simeq O_S \oplus L^*$ and projection formula $\pi_*\pi^*\omega_S \simeq \pi_* O_{\tilde{A}} \otimes \omega_S$, one can show $\omega_S \simeq O_S$.

All K3 surfaces have the same Hodge numbers. By definition, $h^1(S, O_S) = 0$. As K_S is trivial and use Serre duality, $h^2(S, O_S) = h^0(S, K_S) = 1$. Then

$$\chi_S(O_S) = 2.$$

According to Noether's formula,

$$\chi_S(O_S) = 2 = \frac{c_1^2(S) + \chi_{top}(S)}{12}$$

as $c_1(S) = 0$, so

$$\chi_{top}(S) = 24.$$

We also have $\chi_{top}(S) = b_2(S) + 2$. It implies

$$b_2(S) = 22.$$

Thus, we have the full Hodge diamond of K3 surface

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

Let L be an ample line bundle on S. We have $H^1(S,L) = H^2(S,L) = 0$ by Kodaira vanishing theorem, then Riemann-Roch formula shows

$$h^0(S,L) = \chi(S,L) = \frac{1}{2}L^2 + 2$$

When L is very ample, there is an canonical embedding by L

$$\phi_L : S \to \mathbb{P}^{n+1}$$

where $n = \frac{1}{2}L^2$ and $\dim \mathbb{P}(H^0(S,L)^*) = \frac{1}{2}L^2 + 1$. For generic curves in $|L|$, they are smooth irreducible and the restriction of ϕ_L is the canonical map to $\mathbb{P}^n$.

There are some cases when ϕ_L is a double cover and the image is onto a smooth surface with minimal degree n in $\mathbb{P}^{n+1}$ [DEB]:

1. $L^2 = 2$,
2. $L^2 = 8$ and $L = 2D$.

Definition 3.2.6. *The Néron-Severi group $NS(X)$ of an algebraic surface X is the quotient*

$$\mathrm{Pic}(X)/\,\mathrm{Pic}^0(X)$$

where $\mathrm{Pic}^0(X)$ *is the isomorphism class of degree zero line bundles.*

The rank of $NS(X)$ is called Picard number $\rho(X)$ of X.

Definition 3.2.7. $\mathrm{Num}(X)$ *is the quotient*

$$\mathrm{Pic}(X)/\,\mathrm{Pic}^{\tau}(X)$$

by the numerically trivial classes.

Theorem 3.2.8. *[H] For K3 surface S, the natural surjections from the definition are isomorphisms*

$$\mathrm{Pic}(S) \tilde{\rightarrow} NS(S) \tilde{\rightarrow} Num(S)$$

The intersection form on $\mathrm{Pic}(S)$ *is non-degenerate, even, of signature*

$$(1, \rho(S) - 1).$$

3.3 Lagrangian fibrations of irreducible holomorphic symplectic varieties

The theorem of Matsushita reveals that irreducible holomorphic symplectic varieties have special fibration structures. Under some condition, Hwang further showed the base space has to be a projective space.

Theorem 3.3.1. *(Matsushita) [MATSU] Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ with symplectic form σ. If $f : X \to B$ is a proper surjective morphism with connected fibers, onto a normal, projective variety B, $0 < \dim B < 2n$, then*

1. *B is n dimensional,*
2. *generic fiber of f is an n dimensional complex torus,*
3. *any fiber F of f is an n-dimensional Lagrangian sub-variety of X, i.e. $\mathrm{T}\,F \subset \mathrm{T}\,X$ is maximal isotropic under the symplectic structure σ.*

Remark 3.3.2. *For smooth irreducible holomorphic symplectic varieties X, generic fiber of f is an abelian variety. In fact, even if X is not projective, generic fiber of h can be shown to be abelian varieties too.*

Theorem 3.3.3. *(Hwang) In the context of Matsushita's theorem, if B is smooth, then it is isomorphic to $\mathbb{P}^n$. [HW]*

Example 3.3.4. *(Elliptic K3 surface) The lowest dimension example we have is an elliptic K3 surface, i.e. there is a surjection morphism*

$$\pi : S \to \mathbb{P}^1$$

such that a generic fiber is a smooth curve of genus one.

A K3 surface admits an elliptic fibration if and only if there exists a non-trivial line bundle L with self-intersection zero $(L.L) = 0$.

If S is an elliptic K3 surface, then

$$S^{[n]} \to Sym^n(S) \to Sym^n(\mathbb{P}^1) \to \mathbb{P}^n$$

is also a Lagrangian fibration with generic fiber being product of elliptic curves.

Example 3.3.5. *(Beauville-Mukai system) The Beauville-Mukai system is an example of moduli space of sheaves on K3 surface to be discussed in detail in the following sections. Here we state the results without explanation.*

On a K3 surface S, we take a smooth genus g curve C. By Riemann-Roch formula, the complete linear system of C is g dimensional and denote the family by

$$\mathcal{C} \to \mathbb{P}^g.$$

Consider the relative compactified Jacobian $\overline{\mathrm{Jac}^d_H(\mathcal{C}/\mathbb{P}^g)}$ of some degree d [AK].

The support map

$$\overline{\mathrm{Jac}^d_H(\mathcal{C}/\mathbb{P}^g)} \to |C| = \mathbb{P}^g$$

is a Lagrangian fibration. A generic point in $\mathbb{P}^g$ corresponding to a smooth curve, above which the fiber is a g dimensional abelian variety which is the degree d Jacobian of the curve in the base space. They are contained in the smooth locus of the moduli space. Fibers above reducible or non-smooth curves are singular varieties, but the moduli space at those point are possibly smooth.

From another perspective, consider a polarization H of the K3 surface, $\overline{\mathrm{Jac}^d_H(\mathcal{C}/\mathbb{P}^g)}$ is same as the moduli space of H semi-stable sheaves with Mukai vector$v = (0, [C], d + 1 - g(C))$, $\mathcal{M}_{v,H}(S)$. The Lagrangian *fibration is given by the support map*

$$\mathcal{M}_{v,H}(S) \to |C| = \mathbb{P}^g .$$

3.4 Singular irreducible holomorphic symplectic varieties

Irreducible holomorphic symplectic manifolds are rare, and it is worthwhile to extend the notion to singular symplectic varieties. The work of Namikawa demonstrates some progress in this direction [NAS].

Definition 3.4.1. *[BS, F] A normal variety X is holomorphic symplectic if its smooth locus X^{sm} has a holomorphic symplectic 2-form σ, i.e. closed and non-degenerate, and for any resolution of singularity*

$$\gamma : \tilde{X} \to X$$

σ extends to a holomorphic 2-form $\tilde{\sigma}$ on $\tilde{X}$.

Furthermore, if

$$h^{2,0}(\tilde{X}) = 1$$

then X is said to be primitively holomorphic symplectic.

If $\tilde{\sigma}$ is a non-degenerate closed symplectic form on $\tilde{X}$, then $\tilde{X}$ is a symplectic resolution of X.

Definition 3.4.2. *[MT] A V-manifold is a normal algebraic variety with at worst quotient singularities, i.e. it is locally a quotient of an affine space by a finite group.*

Let X be a primitively holomorphic symplectic V-manifold. If $\pi_1(X) = 0$ and X is complete, then X is a singular irreducible holomorphic symplectic variety.

A V-manifold X is holomorphic symplectic if and only if $X - A$ for some analytic subset $A \subset X$ of codimension ≥ 2 is holomorphic symplectic, because any non-degenerate holomorphic 2-form on $X - A$ can be extended to X [F].

Most of our examples derived from relative Prym varieties associated to del Pezzo surfaces are singular holomorphic symplectic varieties which do not admit symplectic resolutions.

Definition 3.4.3. *[MT] Let Y be a closed irreducible sub-variety of a singular holomorphic symplectic variety X with symplectic form σ. It is Lagrangian if*

1. $\dim Y = \frac{1}{2} \dim X$

2. $Y_0 = Y^{sm} \cap X^{sm} \neq \emptyset$ *and* $\sigma|_{Y_0} = 0$

A surjective morphism with connected fiber $f : X \to B$ *from a holomorphic symplectic variety of dimension* $2n$ *to a normal variety* B *with dimension* n *is Lagrangian if the generic fiber of* f *is Lagrangian sub-variety of* X*.*

CHAPTER 4

Moduli space of semi-stable sheaves on K3 surface

We summarize the definition of stability conditions on pure sheaves in the general setting following [HL, S, J2, MAT], and our use-case will be pure dimension one sheaves on K3 surface. Next we highlight the main results of the moduli space of semi-stable sheaves on K3 surfaces. It is endowed with a symplectic structure. When smooth and compact, it is an example of irreducible holomorphic symplectic manifold. From another view-point, the moduli space is the relative compactified Jacobian variety, which admits a Lagrangian fibration in Jacobians under the support map. The construction of the relative Prym variety in later section is built on this.

4.1 Pure sheaves of dimension one on surfaces

Definition 4.1.1. *[HL] Let X be a smooth projective variety of dimension n and let F be a coherent sheaf on X. The support of F is the subscheme of X*

$$Supp(F) = \{x \in X \mid F_x \neq 0\}$$

associated to the ideal sheaf of F, i.e. the kernel of $\mathcal{O}_X \to \mathcal{E}nd(F)$. The dimension of F is the dimension of $Supp(F)$.

Definition 4.1.2. *[HL] A sheaf F is of pure dimension d if for every non-trivial subsheaf $F' \subset F$,*

$$\dim Supp(F') = d.$$

We focus on pure sheaf of dimension one, that is sheaves with no embedded points.

There is another notion of determinantal support.

Let F be a pure sheaf of dimension one, then there exists a length one locally free resolution

$$0 \to L_1 \xrightarrow{f} L_0 \to F \to 0$$

Definition 4.1.3. *[HL] The determinantal support of F is defined as the subscheme of X*

$$supp_{\det} F = \{\det f = 0\}$$

Moreover, it defines the first Chern class of F,

$$c_1(F) = [supp_{\det} F]$$

.

The definition does not depend on the choice of the locally free resolution. Later when we use the word support of a sheaf, we always refer to the determinantal support. It behaves well in families.

4.2 Stability conditions

On a smooth projective variety, we hope to construct a moduli space of sheaves with some topological invariants fixed. In order to have a nice moduli space, we need some notion of boundedness which is guaranteed by taking "stable" sheaves. There are several types of stability conditions, for example, slope stability and Gieseker stability. The slope stability condition does not work for pure dimension one sheaves on a surface, as the rank is zero. We choose to adopt the Gieseker stability, which avoids this problem by considering the Hilbert polynomial of the sheaf.

Let (X, H) be a smooth projective surface with a polarization i.e. an ample bundle H. F is a pure sheaf on X of dimension d.

The Euler characteristic of F is

$$\chi(X, F) = \sum_i (-1)^i \dim h^i(X, F)$$

the Hilbert polynomial of F is a polynomial of m

$$P_H(F,m) = \chi(F \otimes O_X(m)) = \sum_{i=0}^{d} a_i(F)\frac{m^i}{i!}$$

We know the degree of the Hilbert polynomial is the dimension of the support of F and the coefficient a_d is telling the rank of F.

Definition 4.2.1. *The Gieseker stability or Hilbert polynomial stability is that F is H (semi)-stable if for any pure subsheaf $E \subset F$*

$$p_H(E,m) < (\leq) p_H(F,m)$$

for $m \gg 0$, where

$$p_H(F,m) = \frac{P_H(F,m)}{a_d}.$$

In the case of pure dimension one sheaf, because

$$P_H(F,m) = \chi(F \otimes O_X(m)) = \chi(F) + (c_1(F).H)m$$

this stability condition is equivalent with a new slope stability.

Let

$$\mu_H(F) = \frac{\chi(F)}{c_1(F).H}$$

be the slope of pure dimension one sheaf F with respect to polarization H, then

Definition 4.2.2. *A pure sheaf of dimension one F is H (semi)-stable if for any non-trivial subsheaf E,*

$$\mu_H(E) < (\leq) \mu_H(F)$$

Or equivalently, if for any quotient sheaf $F \to E$ which is pure of dimension one,

$$\mu_H(E) > (\geq) \mu_H(F)$$

When equality holds, the sheaf is called strictly H semi-stable.

Remark 4.2.3. *If F is a stable sheaf then the only non-trivial endomorphism of F is isomorphism, otherwise* *there is a subsheaf of equal slope. Thus $Hom(F, F) = \mathbb{C}$, F is a simple sheaf.*

Remark 4.2.4. *Any pure sheaf of dimension one and rank one supported on an integral curve is stable with respect to any polarization as there is no non-trivial subsheaf.*

If F is a pure sheaf of dimension one and rank one supported on a non-irreducible curve C, then F is semi-stable if and only if for every subcurve $C' \subset C$

$$\mu_H(F) \leq \mu_H(F_{C'})$$

where $F_{C'} = F|_{C'}/T_0(F|_{C'})$ is pure of dimension one.

Any H semi-stable sheaf has a non-unique filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$$

called the Jordan-Hölder filtration, such that each $gr_i(F) = F_i/F_{i-1}$ is H stable with reduced Hilbert polynomial equal to $p_H(F, m)$. The graded sheaf is unique

$$gr_H(F) := \oplus_{i=1}^{n} gr_i(F)$$

If F is isomorphic to a direct sum of H stable sheaves with reduced Hilbert polynomial equal to $p_H(F, m)$, then F is called H polystable.

Two pure sheaves F and F' are S-equivalent with respect to H if

$$gr_H(F) = gr_H(F')$$

The S-equivalence class of stable sheaves contains single element.

4.3 Moduli space of semi-stable sheaves on K3 surfaces

There is a general theory on moduli space of stable sheaves on a projective variety, yet we will focus on the case of pure sheaf of dimension one on a K3 surface.

Definition 4.3.1. *Let S be a K3 surface, the Mukai lattice is the even cohomology of S*

$$H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$$

endowed with

$$\langle v, w\rangle = \int_S -v_0 w_2 + v_1 w_1 - v_2 w_0$$

where v, w are called Mukai vectors.

The Mukai vector of a sheaf F on S is

$$\begin{aligned} v(F) &:= \mathrm{ch}(F)\sqrt{\mathrm{Td}_S} = (r(F), c_1(F), \frac{c_1^2(F)}{2} - c_2(F)).(1,0,1) \\ &= (r(F), c_1(F), \frac{c_1^2(F)}{2} - c_2(F) + r(F)). \end{aligned}$$

For a pure sheaf of dimension one, the Mukai vector is

$$v(F) = (0, [C], \chi(F)) = (0, [C], 1 - g + d)$$

where C is the determinantal support of F with genus g, and the degree of F is d.

Let

$$\mathcal{M}_{v,H}(S)$$

be the moduli space of H semi-sheaves of pure dimension one on S with Mukai vector

$$v = (0, [C], \chi(F))$$

It is a remarkable result that $\mathcal{M}_{v,H}$ is a $2g$ dimensional projective variety due to Simpson [SIM]. Furthermore, Mukai showed that the smooth locus containing all stable sheaves $\mathcal{M}^s_{v,H}$ carries a symplectic structure [MUK].

In the general case where the Mukai vector is not necessarily that of a pure sheaf of dimension one, we have

Theorem 4.3.2. *$\mathcal{M}_{v,H}(S)$ is a projective scheme. The closed points parametrize S-equivalence classes of H semi-stable sheaves, that is, isomorphism classes of H polystable sheaves.*

At strictly stable sheaves $[F]$, the S-equivalence class contains a single element as itself, and $\mathcal{M}_{v,H}(S)$ is *smooth. The dimension of $\mathcal{M}_{v,H}(S)$ is*

$$\langle v(F), v(F)\rangle + 2$$

If every semi-stable sheaf is stable, then $\mathcal{M}_{v,H}(S)$ is smooth and deformation equivalent to $S^{[n]}$, where $2n = \langle v(F), v(F)\rangle + 2$.

To understand the tangent space and the symplectic structure, we view the tangent space of $\mathcal{M}_{v,H}$ to a point $[F]$ by deformation theory as

$$\mathrm{Ext}^1(F, F)$$

For $i \geq 0$, there is a trace map

$$tr : \mathrm{Ext}^i(F, F) \to H^i(S, \mathcal{O}_S)$$

and set

$$\mathrm{Ext}^i(F, F)_0 = \ker(tr : \mathrm{Ext}^i(F, F) \to H^i(S, \mathcal{O}_S))$$

The holomorphic symplectic form is defined as following

$$\sigma : \mathrm{Ext}^1(F, F) \times \mathrm{Ext}^1(F, F) \stackrel{\cup}{\to} \mathrm{Ext}^2(F, F) \stackrel{tr}{\cong} H^2(S, \mathcal{O}_S) = H^0(S, K_S)^\vee = \mathbb{C}$$

where $\cup$ is the Yoneda product.

Moreover, if F stable, then it is simple, and $K_S \simeq \mathcal{O}_S$,

$$\mathrm{Ext}^2(F, F) \simeq \mathrm{Hom}(F, F) = \mathbb{C},$$

thus the trace map is isomorphism and $\mathrm{Ext}^2(F, F)_0 = 0$. The obstruction to deforming F is characterized by $\mathrm{Ext}^2(F, F)_0 = 0$, so $\mathcal{M}_{v,H}(S)$ is smooth at F.

To study the local structure of $\mathcal{M}_{v,H}(S)$ at a sheaf F, we state the results using the Kuranishi map from [KLS]. The Kuranishi map is a formal map

$$k : \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)_0$$

such that k is equivariant with respect to the natural conjugation action of $\mathbb{P}(Aut(F))$, $k^{-1}(0)$ is a base of the miniversal deformation of F, the expansion of k is a formal series $k = k_2 + k_3 + \ldots$ where k_2 is a quadratic given by cup product. Then by GIT construction and Luna slice Theorem, $(k_2^{-1}(0)//\,\mathbb{P}(Aut(F)), 0)$ is a local analytic model of $(\mathcal{M}_{v,H}(S), [F])$ [MAT].

For example, let $F = F_1 \oplus F_2$ be a polystable sheaf, where F_i is supported on C_i, C_1, C_2 are two smooth curves meeting in n points. Then $(\mathcal{M}_{v,H}(S), [F])$ is analytically isomorphic to $(\mathbb{C}^{g_1+g_2} \times \bar{\Sigma}, 0)$, where $\bar{\Sigma}$ is the affine cone over a hyperplane section of the Segre embedding $\sigma_{n-1,n-1} : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n^2-1}$ [MAT].

Next we see how the choice of the polarization affects the moduli space.

Definition 4.3.3. *[PR] Let v be a Mukai vector. A polarization H of S is v-generic if for any H semi-stable sheaf F with Mukai vector v and any subsheaf $E \subset F$ satisfying $p_H(F, m) = p_H(E, m)$, Mukai vector of E is collinear with v.*

If the Mukai vector v is primitive, i.e. not divisible in Mukai lattice, and H is a v-generic polarization, then every H semi-stable sheaf is stable and thus $\mathcal{M}_{v,H}$ is smooth. This is because it forces every subsheaf of a strictly semi-stable sheaf with the same reduced Hilbert polynomial to share the same Mukai vector, implying the subsheaf is the sheaf itself.

Definition 4.3.4. *Let v be the Mukai vector of a pure sheaf of dimension one, then call the set of non v-generic polarizations to be W_v, it is locally a finite union of linear subspaces in $Amp(S)$. Each component of $Amp(S) \setminus W_v$ is called a chamber.*

Theorem 4.3.5. *Let v be a primitive Mukai vector of a pure sheaf of dimension one. Let H be a non v-generic polarization, and H' lies on the chamber next to H. There is a birational morphism which is a resolution of singularities*

$$\mathcal{M}_{v,H'} \overset{bir}{\to} \mathcal{M}_{v,H}$$

which is an isomorphism on the sheaves supported on integral curves.

4.4 Relative compactified Jacobian

Using the moduli space construction discussed before, the Beauville-Mukai system is

$$\mathcal{M}_{v,H}(S)$$

where $v = (0, [C], d + 1 - g)$ and C is a smooth curve on the K3 surface polarized by H. The support map gives the Lagrangian fibration in Jacobian

$$\pi : \mathcal{M}_{v,H} \to |C| = \mathbb{P}^g$$
$$F \mapsto Supp_{\det}(F)$$

Above a generic point in the base, which is a smooth curve C' linearly equivalent to C, the fiber is the Jacobian $\mathrm{Jac}^d(C')$ of degree d line bundles on C', which is a g dimensional abelian variety.

If $C' \in |C|$ is not smooth but integral, the fiber is the compactified Jacobian $\overline{\mathrm{Jac}^d(C')}$ of dimension g, consisting of rank one degree d torsion free sheaves on C'.

If $C' \in |C|$ is reduced but not irreducible, the compactified Jacobian $\overline{\mathrm{Jac}_H^d(C')}$ parametrizes S-equivalence class of H semi-stable rank one degree d torsion free sheaves on C'.

For example, in [MAT][S], the situation when C' is the union of two smooth curves can be specified.

Lemma 4.4.1. *Let $C' \in |C|$ be a union of two irreducible curves C_1 and C_2 meeting in n points, where C_i are smooth curves with genus g_i, then*

1. *if $\frac{C_1.H}{C.H}\chi \notin \mathbb{Z}$, then the fiber $\overline{\mathrm{Jac}_H^d(C')}$ has n components of dimension g parametrizing stable sheaves*
2. *if $\frac{C_1.H}{C.H}\chi \in \mathbb{Z}$, then the fiber $\overline{\mathrm{Jac}_H^d(C')}$ has $n-1$ components of dimension g parametrizing stable sheaves and a dimension $g+1-n$ locus of S-equivalence class of strictly semi-stable sheaves of type $[F_1 \oplus F_2]$ where F_i is supported on C_i.*

So $\mathcal{M}_{v,H}(S)$ is also the relative compactified Jacobian

$$\overline{\mathrm{Jac}_H^d(C/\mathbb{P}^g)}$$

discussed in a previous section.

CHAPTER 5

Relative Prym varieties from double covers by K3 surfaces

5.1 Relative Prym varieties from double covers by K3 surfaces

The Beauville-Mukai system when smooth gives an irreducible holomorphic symplectic variety fibered in Jacobians. To find more example, it is natural to look for fibrations by more general abelian varieties, e.g. Prym varieties. A Prym variety normally is defined on a double cover of curves with 0 or 2 ramification points, but we will use a generalized Prym variety in the sense to be defined in the section. We know any principally polarized abelian variety of dimension ≤ 3 is a Jacobian variety, and for dimension ≤ 5 is Prym variety.

Is there a relative Prym construction, which is a relative version of taking the Prym variety of a single curve covering map, that works for a family of curves and gives us new examples of irreducible holomorphic symplectic varieties?

The answer is yes. Markushevich and Tikhomirov first proposed the idea in 2007 [MT]. Later Arbarello, Sacca and Ferretti provided the detail in 2012 [AFS]. The construction replaces the fibers in the Beauville-Mukai system by Prym varieties (as sub-varieties of the Jacobians) while the base space shrinks to a sub-linear system of curves on the K3 surface. In particular, the procedure comes from a symplectic involution on the K3 surface.

Before diving into details, we state some background terminologies on covering maps and the Prym variety of map between curves.

5.1.1 Covering maps

The idea of defining a double cover $\pi : X \to Y$ is to give an "even" divisor B on Y such that there exists a line bundle whose square has a section equivalent to B, $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$. Notice it does not necessarily come from a section of $\mathcal{L}$. We give a rigorous definition of an n-cyclic covering following [BHPV].

A covering map may be used in two different settings, analytic and algebraic. In the first sense, an *analytic* covering refers to a triple (X, Y, π) where X and Y are connected complex manifolds and π is a holomorphic map such that any $y \in Y$ has neighborhood U_y and $\pi^{-1}(U_y)$ is a union of disjoint open sets of X that are mapped isomorphically onto U_y under π.

While in the second sense, X and Y are two connected *normal* varieties and π a finite, surjective proper holomorphic map. In fact, π is a covering between manifolds outside of

$$X' = X \backslash \pi^{-1}(\pi(\operatorname{Sing} X) \cup \operatorname{Sing} Y)$$

and outside of

$$\pi^{-1}(\pi(S)), S = \{x \in X' | \operatorname{rank}(d\pi)_x \leq \dim X - 1\}$$

it is an analytic covering.

The degree of π is the degree of π restricted on $X \backslash \pi^{-1}(\pi(S))$. Properness of π assures that for any $x \in X$ there is a neighborhood U of $\pi(x)$ such that $\pi^{-1}(U) = \cup_{i=1,\dots,n} U_i$, where U_i are disjoint neighborhoods of x. And independent on U, the degree of $\pi | \pi^{-1}(U_1)$ is the *branching order* e_x of π at x. When $e_x \geq 2$, x is a *ramification point* and $\pi(x)$ is a branch point.

A covering with no ramification point is a covering in the first sense, which is called an unramified or an étale covering. Now assume X and Y are smooth varieties, the ramification locus R is defined to be the zero locus of the canonical section in $\operatorname{Hom}(\pi^*(K_Y), K_X)$, and

$$K_X = \pi^*(K_Y) \otimes O_X(R).$$

Lemma 5.1.1. *(Hurwitz formula) If R is the ramification locus of a branched covering and R_i is its irreducible components, e_i is the branching order of some smooth point $x \in R_i$. Then $R = \sum(e_i - 1)R_i$.*

Lemma 5.1.2. *Let $\pi : X \to Y$ be a covering map of degree d between two compact smooth manifolds. If $\mathcal{L}$ is a line bundle on Y which is pulled back to trivial bundle on X, i.e. $\pi^*\mathcal{L} = O_X$, then $\mathcal{L}$ is a torsion line bundle, $\mathcal{L}^{\otimes d} = O_Y$.*

In order to construct Lagrangian fibrations in later sections, we mainly deal with coverings of surfaces with ramifications, so we follow the second definition where X and Y are smooth.

The previous lemma points us to construct a cyclic covering in the following way. Let Y be a compact complex manifold and B be a divisor. If there exists a line bundle $\mathcal{L}$ such that

$$\mathcal{L}^{\otimes n} = \mathcal{O}_Y(B)$$

and a section $s_B \in \Gamma(Y, \mathcal{O}_Y(B))$ with zero locus being B, then we can define the following.

Consider $p : L \to Y$ the projection map from the total space of $\mathcal{L}$. Let η be a tautological section of $\Gamma(L, p^*(\mathcal{L}))$, then the zero locus of

$$p^* s_B - \eta^n$$

inside the total space L defines X as a sub-variety.

When $B \neq 0$ and reduced, X is an irreducible normal analytic subspace of L, and (X, Y, π) is an n-cyclic covering branched over B, determined by $\mathcal{L}$. When $B = 0$, we are taking a torsion line bundle of order n in $\mathrm{Pic}(Y)$, and (X, Y, π) is an n-cyclic étale covering.

If $\mathrm{Pic}(Y)$ is torsion-free, B uniquely determines $\mathcal{L}$ and thus the covering space. The only singularity of X must occur over singularities of B. If B is smooth, X is so too.

We will always use

$$\pi : X \to Y, B$$

to denote a covering map with ramification locus B in this paper. The following is a useful lemma stated in [BHPV].

Lemma 5.1.3. *Let $\pi : X \to Y$ be an n-cyclic covering branched over a smooth divisor B determined by $\mathcal{L} \in \mathrm{Pic}(Y)$. By definition, $\mathcal{L}^{\otimes n} = \mathcal{O}_Y(B)$. Denote the reduced divisor of $\pi^{-1}(B)$ by B_0, then we have $\pi^*(B) = nB_0$ and $K_X = \pi^*(K_Y \otimes \mathcal{L}^{n-1})$ due to the Hurwitz formula.*

5.1.2 Prym variety

We follow the definition in [MUM]. Let π be a double cover of smooth integral curves

$$\pi : C \to C'$$

then there is an involution $\varepsilon : C \to C$ exchanging the two sheets above any point, inducing

$$\varepsilon^* : \mathrm{Jac}^0(C) \to \mathrm{Jac}^0(C)$$

Also, π induces a norm map between the Jacobians

$$\begin{aligned} \mathrm{Nm} : \mathrm{Jac}^0(C) &\to \mathrm{Jac}^0(C') \\ \mathcal{O}_C(p) &\mapsto \mathcal{O}_{C'}(\pi(p)) \end{aligned}$$

for any divisor x on C and divisor y on C'

$$\pi^{-1}(\mathrm{Nm}(x)) = x + \varepsilon^*(x)$$

$$\varepsilon(\pi^*(y)) = \pi^*(y)$$

Therefore ε acts as identity map on $\pi^*(\mathrm{Jac}(C'))$ and as $-id$ on $\ker Nm$.

In fact, the Prym variety of the map is defined to be the subvariety of $\mathrm{Jac}^0(C)$ which is the connected component containing the trivial element in the kernel of the norm map $\mathrm{Nm}_\pi : \mathrm{Jac}^0(C) \to \mathrm{Jac}^0(C')$. It is the same as the connected component containing the trivial element in the fixed locus of $-1 \circ \varepsilon^*$ on JC, where -1 is taking a line bundle to its dual, $-L = Hom_C(L, \mathcal{O}_C) = L^*$

$$\begin{aligned} P(C, \pi) &:= (\ker \mathrm{Nm}_\pi)^0 \\ &= Fix^0(-\varepsilon^*) \\ &= \{x - \varepsilon(x) | x \in C\} \end{aligned}$$

The Prym variety P is often thought of as the "odd" part of $\mathrm{Jac}(C)$.

Let g be the genus of the curve C' and the number of branch points to be $2n$, then genus of C is $2g + n - 1$.

Thus

$$\begin{aligned}
\dim \mathrm{Jac}^0(C) &= 2g + n - 1 \\
\dim \mathrm{Jac}^0(C') &= g \\
\dim P(C, \pi) &= \dim \mathrm{Jac}^0(C) - \dim \mathrm{Jac}^0(C') \\
&= g + n - 1
\end{aligned}$$

Theorem 5.1.4. *The Prym variety $P(C, \pi)$ as a sub-abelian variety inherits a polarization from* $\mathrm{Jac}^0(C)$*, of which the polarization type is*

$$(1, \ldots, 1, 2, \ldots, 2)$$

where there are $g + n - 1$ entries and the number of 2's is g. Only when π is unramified or has only two branch points is $P(C, \pi)$ principally polarized. [MUM]

This is a crucial lemma for our use later. We will see for the relative Prym associated to Enriques surface, a generic fiber in the Lagrangian fibration is a principally polarized variety. This is because the double cover of generic curves is unramified, as the K3 universal cover of an Enriques surface is etalé. In fact, this relative Prym variety is essentially a relative Jacobian fibration.

However, in the relative Prym varieties associated to del Pezzo surfaces, in almost all of the cases, a generic fiber is a Prym variety with non-principal polarization and taking dual is no longer trivial.

Lemma 5.1.5. *If the curve C is hyperelliptic with involution i, and $\pi : C \to C'$ is a double cover with involution τ, then the Prym variety has a double cover by a Jacobian variety*

$$P(C, \pi) \simeq (i \circ \tau)^* J(\bar{C})$$

where $\bar{C} = C/\tau \circ i$ is a smooth curve.

Proof. As any degree zero divisor can be decomposed into the difference of two effective divisors, the involution i induced on $\mathrm{Jac}^0(C)$ is the map $-id$. Thus $P(C, \pi) = Fix^0(-\tau^*) = Fix^0((i \circ \tau)^*) = (i \circ \tau)^*(\mathrm{Jac}(\bar{C}))$, where $\bar{C}$ is the quotient curve by the composition of two involutions. Note $i \circ \tau = \tau \circ i$. $\square$

There is no classical definition of the Prym variety of singular curves to begin with. What we do in the following is to define the relative Prym variety for a family of double cover of curves and study its specialization on singular curves [AFS]. As for the Jacobian variety of singular curves, we should refer to [KS].

5.2 Relative Prym variety associated to Enriques surface

The idea of the construction in this section was proposed by Markushevich and Tikhomirov in [MT], which is to take some subspace of the moduli space of sheaves $\mathcal{M}$ or the relative Jacobian $\mathcal{J}$ through the fixed locus of some involution. In particular, when the K3 surface admits an anti-symplectic involution, it induces an anti-symplectic involution on $\mathcal{J}$.

5.2.1 Symplectic quotient of K3 surfaces

Nikulin gave a complete classification of (S, τ), where S is a K3 surface with an anti-symplectic involution τ using lattice theory in [N]. Let $Y = S/\tau$ be the quotient surface, then there are two cases:

1. If $Fix(\tau) = \emptyset$, then Y is a smooth Enriques surface;
2. If $Fix(\tau) \neq \emptyset$, then Y is a smooth rational surface.

Markushevich and Tikhomirov first worked out the case when Y is a degree 2 del Pezzo surface case. Later, in [AFS], Arbarello, Sacca and Ferretti systematized this approach and presented a rigorous moduli space construction of the "relative Prym variety". The description was based on an example derived from K3 double cover of an Enriques surface. We summarize the key steps in the construction.

5.2.2 Definition

We consider a general Enriques surface T. An Enriques surface is a smooth surface whose square of the canonical bundle is trivial. Its universal cover

$$\pi : S \to T$$

is an unramified double cover associated to the 2 torsion bundle K_T. S has trivial canonical bundle, thus is a K3 surface. We denote the anti-sympletic involution of π by τ.

Consider an irreducible curve $C' \subset T$, and let $C := \pi^{-1}(C')$, C is invariant under τ. Let

$$g(C) = g,$$

$g \geq 3$ odd. Then as C is an unramified cover of C',

$$g(C') = \frac{1+g}{2}$$

and

$$\dim |C'| = \frac{C'^2}{2} = g(C') - 1 = \frac{g-1}{2}.$$

On S,

$$\dim |C| = g$$

the Beauville-Mukai system

$$\overline{Jac^0(C/\mathbb{P}^g)} \to |C|$$

is a Jacobian fibration. It can also be viewed as the moduli space of semi-stable sheaves

$$\mathcal{M}_{v,H}$$

where $v = (0, [C], 1-g)$ and H is a polarization of S.

Under pull-back of π, $|C'|$ corresponds to the sub-linear system in $|C|$ which consists of τ-invariant curves. Denote the sub-linear system of τ-invariant curves by $\mathcal{C}$, we have

$$\begin{tikzcd} \mathcal{C}/\pi^*|C'| \arrow[d, "\pi"] & S \arrow[d, "\pi"] \\ C'/|C'| & T \end{tikzcd}$$

Now π is a relative double cover parametrized by $|C'|$ and the goal is to extend the Prym of a single double cover to a relative version. Just like the definition of a classical Prym variety, $P(C,\tau) = Fix^0(-\tau^*) \subset \mathrm{Jac}^0(C)$, we now take a subspace of the compactified relative Jacobian of the family of curves $|C|$, that is $\overline{\mathrm{Jac}^0_H(C/\mathbb{P}^g)} = \mathcal{M}_{v,H}(S)$, where $v = (0, [C], 1-g)$ and H is a polarization of S.

τ^* is an involution on $\mathcal{M}_{v,H}$, but multiplication by -1 has no meaning now. We replace it by a relative version of -1. For F a sheaf on S inside $\mathcal{M}_{v,H}$, define

$$\sigma(F) = \mathcal{E}xt^1_S(F, \mathcal{O}_S(-C))$$

It satisfies the following:

1. σ^2= id, as pure dimension one sheaves are reflexive. When $F \in \mathrm{Jac}^0(C_0)$, $\sigma(F) \cong Hom_{C_0}(F, O_{C_0})$, so it is a relative version of -1.

2. $v(\sigma(F)) = v(F)$. And if H is taken to be C, then σ is a regular involution on $\mathcal{M}_{v,H}$, because the C semi-stability for F is equivalent with the $\sigma(C) = C$ semi-stability of $\sigma(F)$.

Finally define the relative Prym variety to be

$$\mathcal{P}_{v,H}(S) = \overline{\mathcal{F}ix^0(\sigma \circ \tau^*)} \subset \mathcal{M}_{v,H}(S) = \overline{\mathrm{Jac}^0_H(C)}$$

which is the irreducible component of the closure of the fixed locus of the involution containing the zero section.

When $H = C$, the involution is regular, thus there is no need to take the closure

$$\mathcal{P}_{v,C}(S) = \mathcal{F}ix^0(\sigma \circ \tau^*) \subset \mathcal{M}_{v,C}(S) = \overline{\mathrm{Jac}^0_C(C)}$$

But because this polarization is non-generic, $\mathcal{M}_{v,C}(S)$ is singular. Possible singularities come from sheaves supported on non-integral curves, for example $F = F_1 \oplus F_2$ supported on $C_1 \cup C_2$.

From now on, we take $H = C$ and denote $\mathcal{P} := \mathcal{P}_{v,C}(S)$.

As the fixed locus of a regular involution on a singular space, the singularity of $\mathcal{P}$ is controlled by that of the moduli space $\mathcal{M} := \mathcal{M}_{v,C}(S)$, because the fixed locus of a regular involution on the smooth locus is smooth

$$\mathrm{Sing}(\mathcal{P}) \subset \mathrm{Sing}(\mathcal{M}) = \{ \text{ semi-stable sheaves F supported on non-integral curves } \}.$$

If there is a symplectic resolution $r : \hat{\mathcal{P}} \to \mathcal{P}$ such that $\hat{\mathcal{P}}$ has a unique non-degenerate symplectic form, then we have a new example of a smooth irreducible holomorphic symplectic variety fibered by Prym varieties.

5.2.3 Properties

Theorem 5.2.1. *[AFS] When $|C|$ is non-hyperelliptic, $\mathcal{P}$ is a singular irreducible holomorphic symplectic variety. In particular, the symplectic singularity is not resolvable because locally it is analytically* $\mathbb{C}^{2n}/\pm 1$

for $n \geq 2$. Any desingularization $\tilde{\mathcal{P}}$ is simply-connected and $h^{2,0}(\tilde{\mathcal{P}}) = 1$.

When $|C|$ is hyperelliptic, $\mathcal{P}$ has a symplectic resolution which is birational to a relative compactified Jacobian $\mathcal{M}_{v',H'}(\hat{S})$ of type $\hat{S}^{[g-1]}$, where $\hat{S}$ is a another K3 surface, it is the minimal resolution of $S/i \circ \varepsilon$, where i is the hyperelliptic involution induced by $|C|$ on the K3 surface.

Remark 5.2.2. *In the case when $|C|$ is hyperelliptic, $\mathcal{P}$ is a Prym fibration, but it is birational to a Jacobian fibration. This is because in this special case, $S \to T$ is an unramified double cover, and the Prym variety of the double cover of curves has twice the principal polarization, so it is essentially a Jacobian.*

Irreducible holomorphic symplectic manifolds are indeed rare. The following examples associated to del Pezzo surfaces are all singular holomorphic symplectic varieties.

5.3 Relative Prym variety associated to degree 2 del Pezzo surface

As we see in the classification by Nikulin, the quotient of the K3 surface Y can also be rational surfaces, such as del Pezzo surfaces. Although the double covering π has to be ramified in these cases, when degree of the del Pezzo surface is small, it is worthwhile to study the corresponding relative Prym variety.

Using similar formulation, degree 2 del Pezzo surface case were explored by Markushevich and Tikhomirov in [MT], Menet in [MEN] and degree 3 del Pezzo case was studied by Matteini in [MAT].

Before we describe these systems, it is useful to list some background on del Pezzo surfaces. References used are [DPT, DOL].

5.3.1 Del Pezzo surfaces

Definition 5.3.1. *A del Pezzo surfaces T is a smooth projective surface whose anti-canonical bundle $-K_T$ is ample. The degree of the del Pezzo surface d is the self-intersection of the canonical class.*

Remark 5.3.2. *Some resources use a weaker definition which allows mild singularity, but we will only treat smooth ones.*

We can check del Pezzo surfaces are rational using the Castelnuovo's criterion. When T is a del Pezzo surface, $2K_T$ is not effective, so there is no holomorphic section

$$H^0(T, 2K_T) = 0$$

For $H^0(T, \mathcal{O}_T)$, rewriting it as $H^0(T, K_T - K_T)$, because $-K_T$ is ample, it vanishes by Kodaira vanishing theorem.

Del Pezzo surfaces are classified into 9 classes, with $1 \leq d \leq 9$. Every degree d del Pezzo surface can be geometrically described as $\mathbb{P}^2$ blown up at $9 - d$ points in general position, except in degree 8 it could also be $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.3.3. *Using the anti-canonical ring $\bigoplus_{m\geq 0} H^0(T, -mK_T)$, one can get the following description of the del Pezzo surfaces according to their degree.*

1. *For del Pezzo surfaces with degree $d \geq 3$, the anti-canonical bundle is very ample and the anti-canonical map embedds it in $\mathbb{P}^d$ as a degree d surfaces. $\varphi_{-K_T} : T \hookrightarrow \mathbb{P}^d$.*

2. *For del Pezzo surface with degree $d = 2$, the anti-canonical bundle is base-point free and represents T as a double cover of the projective plane.*

3. *For del Pezzo surface with degree $d = 1$, the bi-anticanonical bundle is base-point free and it represents T as a double cover of a quadric cone in $\mathbb{P}^3$.*

These geometric model will help us to relate the geometric maps in later sections. Also we will take some pluricanonical linear systems on del Pezzo surfaces as the base space for the compact Lagrangian fibration, so we need the following lemma for convenience.

Lemma 5.3.4. *Let T be a del Pezzo surface, then the dimension of the linear system $|-mK_T|$, $m \geq 0$ is*

$$\frac{m(m+1)}{2}\deg(T)$$

Proof. This is a direct computation by the Riemann-Roch formula,

$$\chi(T, \mathcal{O}_T(-mK_T)) = \chi(T, \mathcal{O}_T) + \frac{(-mK_T).(-mK_T - K_T)}{2}$$

Because $-K_T$ is ample, second cohomology of $\mathcal{O}_T(-mK_T)$ vanish under the Serre duality. $H^1(T, \mathcal{O}_T(-mK_T))$ is zero by Kodaira vanishing theorem, so only $H^0(T, \mathcal{O}_T(-mK_T))$ is left. Since T is a blow up of projective plane, $\chi(T, \mathcal{O}_T) = 1$, so $h^0(T, \chi(T, \mathcal{O}_T)) = 1 + \frac{m(m+1)}{2}\deg(T)$. □

Now we come back to the relative Prym $\mathcal{P}$ associated to degree 2 del Pezzo surface considered by Markushevich and Tikhomirov in [MT]. Menet further showed the "dual" of such a Lagrangian fibration is also from a degree 2 del Pezzo surface.

5.3.2 Definition

Let q_4 be a smooth quartic curve in $\mathbb{P}^2$, it determines a double cover

$$\mu : T \to \mathbb{P}^2, q_4$$

branched in q_4. As

$$K_T \simeq \mu^*(K_{\mathbb{P}^2} \otimes O_{\mathbb{P}^2}(\frac{1}{2}q_4))$$

$-K_T$ is ample and $K_T^2 = 2$, so T is a del Pezzo surface of degree 2. μ is in fact the anticanonical map φ_{-K_T} of T.

Let $\mu^{-1}(q_4) = \bar{q}_4$ and $\bar{\Delta}$ be another smooth quartic completely tangent to q_4 in 8 points. $\mu^{-1}(\bar{\Delta}) = \Delta + i(\Delta)$, where Δ is a smooth curve on T and i is the involution on T associated to μ. It can be proved that $\Delta \in |-2K_T|$ and let

$$\rho : S \xrightarrow{2:1} T, \Delta$$

be the double cover branched along Δ and denote by τ the involution on S. By adjunction formula S is a K3 surface.

We consider a linear system $|-K_T|$ on the del Pezzo surface and consider its pull back linear system on S

$$\rho^*|-K_T| = \mathbb{P}^2$$

It contains elements in $|\rho^*(-K_T)|$ invariant under τ. We take a smooth curve $C \in \rho^*|-K_T|$, $g(C) = 3$. Consider polarization of S by C. S is embedded in $\mathbb{P}^3$ as a quartic surface by $\varphi_{\rho^*(-K_T)}$.

$$\begin{tikzcd}
C/\rho^*|-K_T| = \mathbb{P}^2 \subset \arrow[d] & S \arrow[d, "\rho"] \\
C'/|-K_T| = \mathbb{P}^2 \subset & T, \Delta
\end{tikzcd}$$

Let $\mathcal{M}_{v,C}$ be the moduli space of semi-stable sheaves $\mathcal{F}$ on S under polarization C with Mukai vector

$v = (0, C, 1 - g(C))$.

The involution τ on S induces an anti-symplectic involution τ^* on $\mathcal{M}_{v,C}$. Define another anti-symplectic involution by

$$\sigma : \mathcal{F} \mapsto \mathcal{E}xt^1_S(\mathcal{F}, \mathcal{O}_S(-C))$$

Then define the relative Prym variety to be the component in the fix locus of this involution on $\mathcal{M}_{v,C}$ containing the zero-section

$$\mathcal{P}_{v,C} = Fix^0(\sigma \circ \tau^*) \subset \mathcal{M}_{v,C}.$$

5.3.3 Properties

Theorem 5.3.5. *[MT] The relative Prym variety $\mathcal{P}_{v,C}$ is a 4-dimensional singular irreducible holomorphic symplectic variety. The support map*

$$\mathcal{P}_{v,C} \to \rho^*|-K_T| = \mathbb{P}^2$$

is a Lagrangian fibration and generic fibers are abelian surfaces of polarization type $(1,2)$. $\mathcal{P}_{v,C}$ has 28 isolated singularities locally analytically isomorphic to $\mathbb{C}^4 / \pm 1$.

The singular locus of $Sing(\mathcal{P})$ is supported on reducible curves in $\rho^*|-K_T|$. The 28 isolated singularities are from the 28 bitangent lines of the quartic q_4 in $\mathbb{P}^2$. At these points, $C = C_1 \cup C_2$, C_1,C_2 are two irreducible components and they meet at 4 points. Using the Kuranishi map to study the local analytical model

$$T_{\mathcal{F}}\mathcal{P} = \mathbb{C}^4 / \pm 1$$

where $\mathcal{F} = F_1 \oplus F_2$, where F_1, F_2 are supported on C_1 and C_2. We know for symplectic singularities of type $\mathbb{C}^{2n} / \pm 1$, it has symplectic resolution only when $n = 1$, so $\mathcal{P}$ has no symplectic resolution.

Moreover, the birational model of $\mathcal{P}_{v,C}$ is a double quotient of the Hilbert scheme $S^{[2]}/l$

$$\mathcal{P}_{v,C} \overset{bir}{\simeq} S^{[2]}/l$$

where $l = l_0 \circ \tau^*$. τ^* is the induced involution on $S^{[2]}$ by τ on S and l_0 is Beauville's involution for degree 4

K3 surface in $\mathbb{P}^3$

$$l_0 : S^{[2]} \to S^{[2]}$$
$$\xi \mapsto (\langle \xi \rangle \cap S) - \xi$$

where $\langle \xi \rangle$ represents the line spanned by the two elements in ξ in $\mathbb{P}^3$.

Remark 5.3.6. *This result indicates the involution σ on $\mathcal{M}_{v,C}$ defined previously is locally induced by Beauville's involution.*

Remark 5.3.7. *More generally, for any K3 surface S of degree $2r$ in $\mathbb{P}^{r+1}$, the Hilbert scheme of r points $S^{[r]}$ admits a nontrivial birational automorphism. A generic element z in $S^{[r]}$ is r distinct points, which span a codimension 2 linear subspace $\langle z \rangle$ of $\mathbb{P}^{r+1}$ intersecting with S in $2r$ points, so we can define the automorphism l_0 by sending z to the residual intersection $\langle z \rangle \cap S - z$ [BK].*

This map will come up again in the next section for degree 3 del Pezzo surface, where the K3 surface S satisfies the requirement. It will still be a key element in the proof, although the involution is no longer regular. However, in degree 1 del Pezzo case, there is no such involution, but the hyperelliptic involution of the K3 surface becomes a substitution.

Theorem 5.3.8. *[MT] The fixed locus of l on $S^{[2]}$ is the union of a K3 surface Σ and 28 points. Let $\bar{\Sigma}$ be the image of Σ under the quotient $S^{[2]} \to S^{[2]}/l$.*

Let M' be the blow up of M along $\bar{\Sigma}$. Then M' is a singular irreducible holomorphic symplectic variety with 28 singular points of type $\mathbb{C}^4 / \pm 1$.

Moreover, the birational map between M' and $\mathcal{P}_{v,C}$ is a Mukai flop with two centers in both space as Lagrangian sub-varieties isomorphic to $\mathbb{P}^2$.

Using this birational model and the fact that $\mathcal{P}$ only has quotient singularities, one can verify $\pi_1(\mathcal{P}) = 0$ and $h^{2,0}(\tilde{\mathcal{P}}) = 1$ for any desingularization $\tilde{\mathcal{P}}$.

5.3.4 Dual fibration

A following interesting result is that one can construct a "dual" fibration which is from a degree two del Pezzo surface too. The construction relies on a fact due to Pantazis [P], namely, two double covers of curves derived in a certain way have dual Prym varieties.

Pantazis's construction:

Start with a tower of double covers between curves

$$D \xrightarrow{\rho} D_0 \xrightarrow{\pi} \mathbb{P}^1$$

then construct a curve $D' \to \mathbb{P}^1$ in the following way. Above $t \in \mathbb{P}^1$ its points correspond to different ways of lifting a pair of points $\pi^{-1}(t)$ to D, i.e.

$$D' = \{L \in \mathrm{Pic}^{(2)}(D) | \, \mathrm{Nm}(L) = \pi^* O_{\mathbb{P}^1}(1)\}$$

D' has an involution of sending L to the complementary lift L' satisfying $\mathrm{Nm}(L) = \mathrm{Nm}(L')$.

Let the quotient of D' be D'_0. D'_0 has a 2 to 1 cover to $\mathbb{P}^1$ by sending the lifts back to t.

The double cover

$$D' \xrightarrow{\rho'} D'_0 \xrightarrow{\pi'} \mathbb{P}^1$$

is called bigonally related to D.

Theorem 5.3.9. *[P] Under Pantazis's construction, Prym(D/D_0) and Prym(D'/D'_0) are dual abelian varieties.*

Menet gives a concrete way to realize the bigonally related double cover of the hyperplane sections of $S \to T \to \mathbb{P}^2$

$$C_t \to C'_t \to \mathbb{P}^1 .$$

On $\mathbb{P}^2$, one can take a smooth quartic q'_4 bitangent to q_4. The double cover of $\mathbb{P}^2$ branched along q'_4 is another degree 2 del Pezzo surface T'. Denote the double covering by μ'. $\mu'^{-1}(q_4) = \Delta' + i(\Delta')$. Taking Δ' to be the branch locus on T', we get a double cover ρ' of T' by another K3 surface S'

$$S \xrightarrow{\rho} T \xrightarrow{\mu} \mathbb{P}^2, q_4$$

$$S' \xrightarrow{\rho'} T' \xrightarrow{\mu'} \mathbb{P}^2, q'_4$$

The hyperplane sections of $S' \to T' \to \mathbb{P}^2, q'_4$ are bigonally related to the hyperplane sections $C_t \to C'_t \to \mathbb{P}^1$ of $S \to T \to \mathbb{P}^2, q_4$.

Using the duality proved by Pantazis, we have

Proposition 5.3.10. *[MEN] Let the relative Prym varieties associated to the two double cover and linear systems $|\rho^*\mu^*H|$ and $|\rho'^*\mu'^*H|$ be $\mathcal{P}$ and $\mathcal{P}'$, then the Lagrangian fibrations*

$$\begin{tikzcd} \mathcal{P} \arrow[dr] & & \mathcal{P}' \arrow[dl] \\ & \mathbb{P}^2 & \end{tikzcd}$$

are dual, i.e. for a generic line in $\mathbb{P}^2$, the fibers are dual abelian varieties. Also, the two fibration have the same singular locus.

5.4 Relative Prym variety associated to degree 3 del Pezzo surface

This case was studied by Matteini as the next low dimensional example [MATT].

5.4.1 Definition

Let T be a degree 3 del Pezzo surface, take a smooth curve Δ in $|-2K_T|$, as the branch locus, it determines a double cover with involution τ

$$\rho : S \to T, \Delta.$$

Moreover, $-K_T$ embeds T in $\mathbb{P}^3$ as a cubic surface and $\rho^*(-K_T)$ embeds S in $\mathbb{P}^4$ as the intersection of a quadric 3-fold and a cubic cone.

Take a generic smooth curve $C \in \rho^*|-K_T| = \mathbb{P}^3$, $g(C) = 4$. Consider polarization of S by C. $C' = \rho(C)$ is a smooth curve of genus 1.

$$\begin{tikzcd} C/\rho^*|-K_T| = \mathbb{P}^3 \subset \arrow[d] & S \arrow[d, "\rho"] \\ C'/|-K_T| = \mathbb{P}^3 \subset & T, \Delta \end{tikzcd}$$

Let $\mathcal{M}_{v,C}$ be the moduli space of semi-stable sheaves $\mathcal{F}$ on S under polarization C with Mukai vector $v = (0, [C], 1 - g(C))$.

The involution τ on S induces an anti-symplectic involution τ^* on $\mathcal{M}_{v,C}$. Define another anti-symplectic involution by

$$\sigma : \mathcal{F} \mapsto \mathcal{E}xt^1_S(\mathcal{F}, \mathcal{O}_S(-C))$$

The relative Prym variety associated to the linear system $|C| = \rho^*| - K_T|$ on S under polarization C is the component in the fix locus of this involution on $\mathcal{M}_{v,C}$ containing the zero-section

$$\mathcal{P}_{v,C} = Fix^0(\sigma \circ \tau^*) \subset \mathcal{M}_{v,C}$$

5.4.2 Properties

Theorem 5.4.1. *[MAT] The resulting relative Prym variety $\mathcal{P}$ is a 6-dimensional symplectic variety. Generic fibers of the Lagrangian fibration*

$$\mathcal{P} \to \rho^*| - K_T| = \mathbb{P}^3$$

are (1,1,2)-polarization type abelian varieties. $\mathcal{P}$ has 27 $\mathbb{P}^1$ family of $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$-type singularities, and 45 points of isolated singularity of type $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

The analysis of the singularity comes from examining the non-integral curves of $\rho^*| - K_T|$ on S, where the strictly semi-stable sheaves are supported on. The non-integral elements come from the section passing through any of the 27 lines on the cubic.

Matteini further computed the form of the polystable sheaves. A smooth point in Sing($\mathcal{P}$) is locally isomorphic to $(\mathbb{C}^4/\pm 1) \times \mathbb{C}^2$, corresponding to a polystable sheaf of type

$$\mathcal{F} = \mathcal{O}_{C_1}(-2) \oplus \mathcal{F}', \mathcal{F}' \in J^{-2}(C_2)$$

where C_1 is a smooth rational curve and C_2 is a genus 1 curve. A singular point of Sing($\mathcal{P}$) is locally $\mathbb{C}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, corresponding to a polystable sheaf

$$\mathcal{F} = \mathcal{O}_{C_1}(-2) \oplus \mathcal{O}_{C_2}(-2) \oplus \mathcal{O}_{C_3}(-2)$$

C_i are rational curves intersecting each other transversely in pairs in 2 points.

To show the birational model of $\mathcal{P}$, start from a dominant rational map

$$\phi : S^{[3]} \dashrightarrow \mathcal{P}$$
$$D \mapsto D - \tau(D)$$

This is well-defined, because for a generic point $D \in S^{[3]}$, it is three distinct points. Its image on T under ρ determines a hyperplane in $|-K_T|$, whose pull back on S is a τ-invariant curve C, and $D - \tau(D) \in P(C, \tau)$.

Lemma 5.4.2. *For the K3 surface S involved, there exists Beauville's involution on $S^{[3]}$*

$$l_0 : S^{[3]} \dashrightarrow S^{[3]}$$
$$\xi \mapsto (\langle \xi \rangle \cap S) - \xi$$

which is antisympletic.

Theorem 5.4.3. *[MAT] The morphism ϕ is in fact the map induced by involution $l = l_0 \circ \tau^*$. Hence*

$$\mathcal{P} \overset{bir}{\simeq} Bl(S^{[3]})/l,$$

$Bl(S^{[3]})$ is the blow up of $S^{[3]}$ along the indeterminacy locus of l_0. By this birational model, $\pi_1(\mathcal{P}) = 0$, and for any desingularization $\tilde{\mathcal{P}}$, $h^{2,0}(\tilde{\mathcal{P}}) = 1$. $\mathcal{P}$ is a singular irreducible holomorphic symplectic variety.

Remark 5.4.4. *This result indicates the involution σ on $\mathcal{M}$ defined previously is locally induced by Beauville's involution.*

Lastly,

Theorem 5.4.5. *[MAT]* $\chi(\mathcal{P}) = 2283$.

This is by examining the stratification of the discriminant locus of the base space $\rho^*|-K_T|$ of the support map

$$\mathcal{P} \to \rho^*|-K_T| = \mathbb{P}^3$$

The Euler characteristic of $\mathcal{P}$ is decomposed into

$$\sum \chi(S_a)\chi(P_a)$$

where S_a is a point in the stratification and P_a is the fiber over S_a.

CHAPTER 6

Relative Prym variety from degree 1 del Pezzo surfaces

In this section, we present our original work on a new singular holomorphic symplectic variety fibered in Prym varieties of a new polarization type.

We analyze the relative Prym variety $\mathcal{P}$ associated to the double cover from a K3 surface S to a degree 1 del Pezzo T and consider the linear system $|-2K_T|$. For the construction, we take over the systematized approach developed in [AFS]. In particular, we give geometric properties of the surface maps and the linear systems of curves involved. Then we determine the singularities of $\mathcal{P}$. In the end, by finding a rational dominant map from a Beauville-Mukai system on another K3 surface $\hat{S}$ to the relative Prym variety $\mathcal{P}$, we managed to show $H^0(\tilde{P}, \Omega^2_{\tilde{P}}) = \mathbb{C}$ for any desingularization $\gamma : \tilde{\mathcal{P}} \to \mathcal{P}$. Moreover, under some assumption, the fundamental group of $\mathcal{P}$ is the same as that of a quotient space of a simply-connected space by a group at most of order 2.

6.1 Definition

We consider when the K3 surface S has an anti-symplectic involution τ such that the quotient S/τ is a degree 1 del Pezzo surface.

Degree 1 del Pezzo surfaces are isomorphic to $\mathbb{P}^2$ blown up at 8 points. Any 4 points in $\mathbb{P}^2$ of general position can be fixed by a projective transformation, so there is an 8-dim family of degree 1 del Pezzo surface.

Let T be a degree 1 del Pezzo surface and take a smooth curve $\Delta \in |-2K_T|$. As the branch locus, Δ determines a double cover,

$$\pi : S \xrightarrow{2:1} T, \Delta$$

Since $K_S \simeq \pi^*(K_T \otimes O_T(\frac{1}{2}\Delta)) \simeq O_S$, S is a K3 surface. Denote by τ the anti-symplectic involution on S compatible with this double cover.

We start from the linear system $|-2K_T|$ on T. Under the pull-back of π, it corresponds to a sub-linear system of $|\pi^*(-2K_T)|$ on S containing curves which are invariant under involution τ.

Let

$$C' \in |-2K_T|$$

be a smooth integral curve and

$$C := \pi^{-1}(C')$$

Then $\pi^*|-2K_T|$ is the subspace of the complete linear system $|C|$ containing elements that are invariant under the involution τ. Or if we start from the 5 dimensional complete linear system $|C|$ on S, there is a 3 dimensional subspace consisting of curves which are invariant under τ and are pull-back of curves in $|-2K_T|$ on T.

By adjunction formula, $2g(C) - 2 = C.(C + K_S)$, $C \in \pi^*|-2K_T|$, so

$$g(C) = 5$$

$2g(C') - 2 = C'.(C' + K_T)$, $C' \in |-2K_T|$, so

$$g(C') = 2.$$

Denote by $\mathcal{C}$ and $\mathcal{C}'$ the family of curves in linear system $\pi^*|-2K_T|$ and $|-2K_T|$, we summarize the maps into the following diagram

$$\begin{array}{ccc} \mathcal{C}/\pi^*|-2K_T| = \mathbb{P}^3 & \subset & S \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{C}'/|-2K_T| = \mathbb{P}^3 & \subset & T, \Delta \end{array}$$

Now we consider the moduli space of sheaves on the K3 surface S. Consider polarization of S by C, which determines the stability condition, and let

$$\mathcal{M}_{v,C}(S)$$

be the moduli space of C semi-stable sheaves $\mathcal{F}$ on S with Mukai vector

$$v = v(\mathcal{F}) = (0, [C], 1 - g(C))$$

it contains sheaves $\mathcal{F}$ of rank 0 on S supported on a curve in the complete linear system $|C| = \mathbb{P}^5$ and having Euler characteristic $1 - g(C) = -4$, that is to say, the Hilbert polynomial of $\mathcal{F}$ is the same as that of a degree zero line bundle of C pushed forward onto S by inclusion. By previous discussion, this a Jacobian fibration. A generic fiber is a 5 dimensional Jacobian variety supported on smooth integral curves, while there exists polystable sheaves supported on reduced and reducible curves as possible singularities of the whole space.

Theorem 6.1.1. *$\mathcal{M}_{v,C}(S)$ has symplectic structure on the smooth locus. Any stable sheaves are inside the smooth locus. The support map*

$$\mathcal{M}_{v,C}(S) \to |C| = \mathbb{P}^5$$

is a Lagrangian fibration, a generic fiber is the Jacobian of a smooth integral genus 5 curve in $|C|$.

Due to the non-generic choice of the polarization, the moduli space $\mathcal{M}_{v,C}$ is singular, but choosing polarization to be C ensures the involution (specified later) to be regular and will let us have control of the singular locus of the relative Prym variety. In general, when the Mukai vector is primitive, changing the polarization yields a resolution. However, as $|C| = \pi^*| - 2K_T| = 2C_0$ and $g(C)$ is odd, Mukai vector $v = (0, [2C_0], -4)$ is non-primitive, $\mathcal{M}_{v,C}$ is singular along the locus of strictly semi-stable sheaves and yet changing the polarization won't give a symplectic resolution [OG].

Surprisingly, in this case Rapagnetta showed that $\mathcal{M}_{v,C}$ admits a symplectic desingularization achieved in a way originated from O'Grady, where he constructed his example of a smooth 10 dimensional deformation type of irreducible holomorphic symplectic variety OG(10). It is a symplectic resolution of $\mathcal{M}_{v',H}(S)$, where $v' = (2, 0, -2)$ and H is a degree 2 polarization of a deformed K3 surface as a double plane [OG, PR, J1, KLS].

A quick way to see the relation is that we can show

$$\mathcal{M}_{v,C}(S) \overset{bir}{\simeq} \mathcal{M}_{v',H}(S)$$

where $v' = (2, 0, -2)$, $v = (0, [2C_0], -4)$. Because for a generic $\mathcal{E}$ in $\mathcal{M}_{v',H}(S)$, using Riemann-Roch formula,

$\dim H^0(S, \mathcal{E}(1)) = 2$, so let the two holomorphic sections be s_1, s_2, then $\mathcal{E}$ corresponds to exact sequence

$$0 \to O_S(-1) \oplus O_S(-1) \stackrel{s_1,s_2}{\to} \mathcal{E} \to \mathcal{F} \to 0$$

$v(O(-1)) = (1, -1, 1)$, hence $v(\mathcal{F}) \in \mathcal{M}_{v,H}$.

We should notice that our K3 surface is more special than the K3 surface in O'Grady's example, because S possesses another involution which gives the double cover to the del Pezzo surface. Thus, in $\mathcal{M}_{v,C}(S)$ there is some additional singular locus.

Now we start to define the relative Prym variety. We introduce the relative version of " $-$ 1" on $\mathcal{M}_{v,C}$ by

$$\sigma : \mathcal{F} \mapsto \mathcal{E}xt^1_S(\mathcal{F}, O_S(-C))$$

We can show this is a regular involution and is anti-symplectic. At the same time, the involution τ on S induces a regular involution τ^* on $\mathcal{M}_{v,C}$ because the polarization C is invariant under τ. Thus, $\sigma \circ \tau^*$ is a regular involution on $\mathcal{M}_{v,C}$ and preserves the symplectic structure

$$\sigma \circ \tau^* : \mathcal{M}_{v,C} \to \mathcal{M}_{v,C}$$

Definition 6.1.2. *The relative Prym variety associated to the linear system $\pi^*|-2K_T|$ on S under polarization C is the component of the fix locus of this involution on $\mathcal{M}_{v,C}$ containing the zero-section*

$$\mathcal{P}_{v,C} = Fix^0(\sigma \circ \tau^*) \subset \mathcal{M}_{v,C}$$

Theorem 6.1.3. *The relative Prym variety $\mathcal{P}_{v,C}$ is a 6-dimensional projective variety with a symplectic form on the smooth locus. A generic fiber of the Lagrangian fibration*

$$\mathcal{P}_{v,H} \to |C'| = \mathbb{P}^3$$

is an abelian 3-fold of polarization type $(1, 2, 2)$.

Proof. This essentially follows from the main result of [AFS] theorem 3.12 . On a generic point in C', there is a double cover of two irreducible curves with genera 5 and 2, so the Prym variety is an abelian 3-fold. From previous section, we know it's polarization type is $(1, 2, 2)$. $\square$

We are interested in the singularity of $\mathcal{P}_{v,C}$ and whether it has any symplectic resolution.

It is natural to wonder whether the involution $\sigma \circ \tau^*$ will extend to an involution on the desingularization of $\mathcal{M}_{v,C}$ mentioned before, and if so, whether the fix locus gives a symplectic resolution of $\mathcal{P}_{v,C}$. In fact, however, we eventually conclude that $\mathcal{P}_{v,C}$ does not admit symplectic resolution because of its local analytical model.

6.2 Geometry

We describe the double cover $\pi : S \to T$ using equations in weighted projective coordinates, then by looking at the morphism defined by the bi-anticanonical bundle on T, we have a better understanding of reducible members of the linear system $\pi^*|-2K_T|$, which encode the singularities of $\mathcal{P}$ to be discussed in the next section.

Lemma 6.2.1. *Let $\pi : S \to S/\tau = T$, then $\pi^*(-K_T)$ maps S to $\mathbb{P}^2$ as a double cover branched along a sextic curve s_6 in $\mathbb{P}^2$*

$$x_2^6 + x_2^4 f_2 + x_2^2 f_4 + f_6 = 0$$

where f_i is degree i polynomial in $\mathbb{C}[x_0, x_1]$, and S is the zero locus of

$$y^2 = x_2^6 + x_2^4 f_2 + x_2^2 f_4 + f_6$$

in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$ under coordinates $[y : x_0 : x_1 : x_2]$. The involution on S induced by this double cover is $i : y \mapsto -y$.

The involution on S associated to the double cover to T is $\tau : x_2 \mapsto -x_2$. Thus T can be represented in $\mathbb{P}(3, 1, 1, 2)$ by

$$y^2 = x_2^3 + x_2^2 f_2 + x_2 f_4 + f_6$$

Proof. The map onto $\mathbb{P}^2$ is inspired from the results on the relative Prym of degree one del Pezzo surface considering the linear system $\pi^*(-K_T)$ in [MAT]. The representation of T is from [KSC]. $\square$

Meanwhile, the linear system $|-2K_T|$ is not very ample but base-point free [KSC], so consider the following

Lemma 6.2.2. *Let φ_{-2K_T} be the regular morphism given by this linear system $|-2K_T|$. It is a double cover of a quadric cone Λ in $\mathbb{P}^3$ branched over $P \cup \delta$, where $P = [1:0:0:0]$ is the apex of the cone and δ is a sextic curve as the intersection of the cone with a cubic surface c_3 in $\mathbb{P}^3$.*

$$\varphi_{-2K_T} : T \xrightarrow{2:1} \Lambda, P \cup \delta$$

Proof. This essentially comes from examining the anti-canonical ring of T and is mentioned in [DPT] [A]. □

Remark 6.2.3. *Dolgachev had a slightly different description in [DOL]. The map factors through the birational map to the anti-canonical model of T, and is a double cover of the quadric cone only branched over a sextic curve. This is because in [DOL] T is taken to be a weak del Pezzo surface which has A_1 singularities and they are mapped to the apex of the cone.*

Consider a generic hyperplane section in $\mathbb{P}^3$, the ambient space of the quadric cone Λ, its pull-back onto T under φ_{-2K_T} is in $|-2K_T|$. In addition, dimension of the linear system $|-2K_T|$ is 3 according to previous lemma 5.3.4. This implies every curve in $|-2K_T|$ is the pull-back of a hyperplane intersection of the quadric cone Λ. Denote by $\mathcal{H}$ the family of hyperplane sections of Λ, we extend the surface map to a tower of double covers

$$\begin{tikzcd} C \to \pi^*|-2K_T| \arrow[d] & S \arrow[d, "\pi"] & \\ C' \to |-2K_T| \arrow[d] & T, \arrow[d, "\varphi_{-2K_T}"] & \Delta \in |-2K_T| \\ \mathcal{H} \to (\mathbb{P}^3)^\vee & \Lambda \subset \mathbb{P}^3, & \Lambda \cap c_3 = \delta \end{tikzcd} \tag{6.1}$$

where on the left hand side, each space is a 3-dimensional linear system on the corresponding surface, and the vertical arrows represent double cover of curves. A generic hyperplane section of the quadratic cone is $H \cap \Lambda \simeq \mathbb{P}^1$, its pull-back under φ_{-2K_T} is a smooth integral curve C' in $|-2K_T|$, and $\pi^{-1}(C') = C$ is smooth integral in $\pi^*|-2K_T|$.

Using the one to one correspondence between these 3-dimensional linear systems, one can also think of the relative Prym as fibered over the space of hyperplanes in $\mathbb{P}^3$

$$\mathcal{P}_{v,C} \to |C'| = |H| = (\mathbb{P}^3)^\vee$$

One can check Riemann-Roch on those double cover of curves. Because C' intersects with Δ at $\pi^*(-2K_T).(-2K_T) = 4$ points, C is a double cover of C' branched at 4 points. Similarly, hyperplane H in $\mathbb{P}^3$ intersects with δ in 6 points, so C' is a double cover of a rational curve $H \cap \Lambda$ branched at 6 points.

Additionally, we show how to relate the surface maps mentioned so far in one commutative square diagram. We recall a classical result of the projective model of K3 surface.

Lemma 6.2.4. *Let L be an ample line bundle on K3 surface S and $L \simeq O_S(C)$, then either L is very ample, i.e. the morphism φ_L is an embedding with image being degree $2g(C) - 2$ in $\mathbb{P}^{g(C)}$and all curves in $|L|$ are non-hyperelliptic, or in the the following cases φ_L is degree 2 and image of S is of degree $g(C) - 1$*

1. $L^2 = 2$
2. $L^2 = 8$ *and* $L = 2D$ *for some divisor D,* $D^2 = 2$
3. $L.D = 2$ *and* $D^2 = 0$ *for some divisor D*

and thus all curves in $|L|$ are hyperelliptic.

Proof. See [DEB] or [SD]. □

In our case $L = \pi^*(-2K_T)$, it lies in the case (2) and $\varphi_{\pi^*(-2K_T)}$ is a degree 2 map. We decompose $\varphi_{\pi^*(-2K_T)}$ into $v_2 \circ \varphi_{\pi^*(-K_T)}$, where v_2 is the Veronese embedding of degree 2. From previous argument, $\varphi_{\pi^*(-K_T)}$ represents S as a double cover of $\mathbb{P}^2$ branched along a sextic s_6, and v_2 embeds its image in $\mathbb{P}^2$ to $\mathbb{P}^5$ as a quartic surface,

$$\varphi_{\pi^*(-2K_T)} : S \xrightarrow{2:1} \mathbb{P}^2, s_6 \xrightarrow{v_2} V_4 \subset \mathbb{P}^5$$

Let us denote by i the involution on S induced by $\varphi_{\pi^*(-2K_T)}$.

Notice the pull-back of a conic in $\mathbb{P}^2$ is a curve in $|\pi^*(-2K_T)|$, but dimension of the linear system $|\pi^*(-2K_T)|$ is 5, equal to the dimension of conics in $\mathbb{P}^2$. This implies that any curve in $|\pi^*(-2K_T)|$ is hyperelliptic.

Furthermore, this map $\varphi_{\pi^*(-2K_T)}$ on S is compatible with the surface maps on T mentioned before in the following way.

Theorem 6.2.5. *There is a commutative diagram*

$$\begin{tikzcd}
\varphi_{\pi^*(-2K_T)} : & S \arrow[r, "2:1"] \arrow[d, "2:1"] & \mathbb{P}^2 \arrow[r, hook, "v_2"] \arrow[d, "2:1"] & V_4 \subset \mathbb{P}^5 \arrow[d, "2:1"] \\
\varphi_{-2K_T} : & T \arrow[r, "2:1"] & \mathbb{P}(1,1,2) \arrow[r, hook, "i"] & \Lambda \subset \mathbb{P}^3
\end{tikzcd}$$

on the second row, the first map is a 2 to 1 projection

$$[y : x_0 : x_1 : x_2] \mapsto [x_0 : x_1 : x_2]$$

followed by embedding

$$[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_0x_1 : x_1^2 : x_2]$$

Proof. The map on the second row can be found in [CO]. We can define the second and third vertical maps in a desired way to ensure commutativity, i.e.

$$\mathbb{P}^2 \to \mathbb{P}(1,1,2)$$

$$[x_0, x_1, x_2] \mapsto [x_0 : x_1 : x_2^2]$$

and this induces the vertical map on the right. □

6.3 Singularities

The singular locus of $\mathcal{P}_{v,C}$ is contained in the singular locus of $\mathcal{M}_{v,C}$, as the quotient of the smooth locus of $\mathcal{M}_{v,C}$ under a regular involution is smooth. By the theory of moduli space of sheaves, the singular locus of $\mathcal{M}_{v,C}$ is contained in the locus of strictly semi-stable sheaves. Because a torsion free sheaf on an integral curve is stable with respect to any polarization (as there is no non-empty subsheaf), strictly semi-stable sheaves are supported on non-integral curves in $|\pi^*(-2K_T)|$. Moreover, the singular locus of $\mathcal{P}_{v,C}$ lies in the locus of τ invariant curves in $|\pi^*(-2K_T)|$.

The following lemma describes the possible non-integral curves C in $|\pi^*(-2K_T)|$.

Lemma 6.3.1. *Let $\pi : S \to T$ be a double cover from a K3 surface to a del Pezzo surface with involution τ and branch locus Δ. Let C be a τ-invariant curve on S and $\pi : C \to C'$. If C is reducible, then either C' is reducible or C' is completely tangent to Δ, i.e. meeting at tangency points only.*

Proof. In the two cases mentioned, it is possible to cause C to be reducible. The case when C' meets Δ transversally should be excluded because C will be irreducible for topological reasons.[MAT] $\square$

Hence, to have C reducible, its image C' under π has to be reducible or completely tangent to the branch locus Δ. Notice that it is for sure if C' is reducible then C is reducible, but when C' is totally tangent to Δ, we need to analyze specifically as it is possible that C is still irreducible.

Now we examine the following two cases,

1. reducible curves C' in $|-2K_T|$

2. irreducible curves in $|-2K_T|$ completely tangent to Δ

For (1), we look at the image of C' under φ_{-2K_T}, which is a hyperplane section $H \cap \Lambda$ in $\mathbb{P}^3$

$$\begin{tikzcd}
C \in \pi^*|-2K_T| \arrow[d] & S \arrow[d, "\pi"] & \\
C' \in |-2K_T| \arrow[d] & T, \arrow[d, "\varphi_{-2K_T}"] & \Delta \in |-2K_T| \\
H \cap \Lambda, H \in (\mathbb{P}^3)^\vee & \Lambda \subset \mathbb{P}^3, & \Lambda \cap c_3 = \delta
\end{tikzcd} \tag{6.2}$$

and this case decomposes into two possibilities,

(a) $H \cap \Lambda$ is reducible

(b) H is completely tangent to the branch locus sextic curve δ

(a) happens when H passes through the apex P of the quadric cone, which is a $\mathbb{P}^2$ family. The intersection $H \cap \Lambda$ is two rational curves meeting at one point, $L_1.L_2 = 1$. Besides P, they each intersect with δ at 3 points generically, so 4 ramification points of φ_{-2K_T} in total. By Riemann-Hurwitz, their preimage is two genus one curves meeting at one point. On T, generically, they each intersect with branch locus Δ transversally at 2 points, so their preimage on S is two genus 2 curves meeting in 2 points transversally.

Case (b) is described by following lemma

Lemma 6.3.2. *[W] Let $\varphi_{-2K_T} : T \to \Lambda, \delta \cup P$. There is a bijection between two sets of objects, the hyperplanes in $\mathbb{P}^3$ tritangent to δ but not passing through P, and pairs of exceptional curves intersecting at 3 points on del Pezzo surface T, each pair intersecting at 3 points, $\{(E_1, E_2)|E_1.E_2 = 3\}$. There are 120 hyperplanes in $\mathbb{P}^3$ tritangent to δ.*

To consider the preimage of these pair of exceptional curves on S, we know they each intersect with Δ at 2 points generically, so by Riemann-Hurwitz the preimage is two genus zero curves meeting at six points.

For (2), as $C'.\Delta$=4, C' meet with the branch locus Δ at 2 points with multiplicity 2, then $\pi^{-1}(C')$ is two genus 2 curves meeting at 2 points. The possibility that it is a genus 5 irreducible curve with 2 nodal singularities is excluded.

If $C' \in |-2K_T|$ is completely tangent to Δ, then because Δ is in $|-2K_T|$ and any curve in $|-2K_T|$ is the pull-back of some hyperplane section in $\mathbb{P}^3$, denote $\varphi_{-2K_T}(C') = H' \cap \Lambda$, $\varphi_{-2K_T}(\Delta) = H_0 \cap \Lambda$, H' has to be tangent to the conic $H_0 \cap \Lambda$, which results in a quadric cone locus Λ'.

The apex of this quadric cone locus Λ' corresponds to the hyperplane $H' = H_0$, in which case C' coincides with Δ, and the preimage of C' in S is a non-reduced curve 2Σ, where $g(\Sigma) = 2$, as Δ is the branch locus of the double cover $\pi : S \to T$. Also there is a $\mathbb{P}^1$ locus inside Λ' when H' meets Λ in one line with multiplicity 2. In this case, C' is non-reduced, and the preimage of C' in S is also a non-reduced curve 2Σ where $g(\Sigma) = 2$.

In summary, we have

Theorem 6.3.3. *The linear system $\pi^*|-2K_T|$ has one $\mathbb{P}^2$ family, 120 isolated and quadric cone family of reducible or non-reduced curves.*

Proof. The $\mathbb{P}^2$ family is from hyperplanes through the apex of quadric cone (a). The 120 point singularities come from the 120 tritangent planes to the sextic [CKRN] (b). The quadric cone family is from hyperplanes tangent to a conic on the quadric cone (2). □

Using the Kuranishi map, one can achieve the local analytic model of the singularity of the relative Prym variety[AFS], [MAT]. We state the result here.

Theorem 6.3.4. *Let $\pi : S \to T$ be the double cover with anti-symplectic involution τ, C is a τ invariant curve on S. When $C = C_1 \cup C_2$, where C_1, C_2 are two smooth curves, there exists $F = F_1 \oplus F_2$, F_i supported on C_i, a polystable sheaf in the relative Pyrm variety $\mathcal{P}$ defined from C. Let $2k$ be the number of intersection points of C_1, C_2, $\dim(\mathcal{P}) = N$, then $(\mathcal{P}, [F])$ is locally analytically isomorphic to $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k} / \pm 1)$.*

Applying the theorem to previous descriptions of $\pi^{-1}(C')$ in S, we get

Theorem 6.3.5. *The symplectic singularity of the relative Prym variety $\mathcal{P}_{v,C}(S)$ includes*

1. *120 isolated singularities locally analytically isomorphic to $\mathbb{C}^6 / \pm 1$,*
2. *a $\mathbb{P}^2$ family of $\mathbb{C}^4 \times (\mathbb{C}^2 / \pm 1)$-type singularities,*
3. *a quadric cone Λ' family of singularities, generically of $\mathbb{C}^4 \times (\mathbb{C}^2 / \pm 1)$-type, but possibly of worse type in the apex of Λ' and a $\mathbb{P}^1$ locus in Λ'.*

By the theory of symplectic resolutions, as codimension of the 120 point singularities is greater than 4, they are terminal symplectic singularities [Na], so $\mathcal{P}_{v,C}(S)$ admits no symplectic resolution. The $\mathbb{C}^4 \times (\mathbb{C}^2 / \pm 1)$-type singularities can be resolved. In general, symplectic singularities of type $\mathbb{C}^{2n} / \pm 1$ is resolvable only when $n = 1$.

It remains open the singularity of $\mathcal{P}_{v,C}(S)$ at the sheaves which are supported on non-reduced curves corresponding to the apex of Λ' and a $\mathbb{P}^1$ locus in Λ'.

6.4 Hodge number $h^{2,0}(\tilde{\mathcal{P}})$

We show for any desingularization $\tilde{P}$ of the holomorphic symplectic variety $\mathcal{P}_{v,C}(S)$, $h^{(2,0)}(\tilde{P}) = 1$. The main proof relies on a rational dominant map from a Beauville-Mukai system to $\mathcal{P}_{v,C}(S)$.

In Lemma 6.2.4, we mentioned the two anti-symplectic involutions on the K3 surface S, τ associated to the double cover to the del Pezzo surface T and i associated to the hyperelliptic map of $|\pi^*(-2K_T)|$ to $\mathbb{P}^2$.

Lemma 6.4.1. *The composed involution $i \circ \tau$ on S is symplectic with 8 fixed points. The two involutions commute, and the quotient of S under $i \circ \tau$ is a K3 surface $\bar{S}$ with 8 A_1- singularities. Denote the quotient map by $\mu : S \to \bar{S}$. $\bar{S}$ admits a minimal resolution by another K3 surface $\eta : \hat{S} \to \bar{S}$. $\hat{S}$ is an elliptic K3 surface.*

Proof. $i \circ \tau$ acts by $[y : x_0 : x_1 : x_2] \mapsto [-y : x_0 : x_1 : -x_2]$. There are 6 fixed points which are fixed by both τ and i, i.e. $\{y = x_2 = 0\} \cap S$, and 2 points that each are interchanged by τ and i, i.e. $\{x_0 = x_1 = 0\} \cap S$. Thus $\bar{S}$ is a singular K3 surface with 8 rational double points. It's well-known it has a resolution by a K3 surface. Consider $C_0 \in \pi^*|-K_T|$, the strict transform of $\mu(C_0)$ in $\hat{S}$ gives the elliptic fibration, [MAT]. □

So we have

$$\begin{tikzcd}[column sep=small] \varphi_{\pi^*(-2K_T)} : & S \arrow[r, "2:1"] \arrow[d, "2:1"] \arrow[dr, "\mu"] & \mathbb{P}^2 \arrow[r, hook, "\nu_2"] & V_4 \subset \mathbb{P}^5 \\ \varphi_{-2K_T} : & T & \bar{S} & \hat{S} \arrow[l, "\eta"] \end{tikzcd}$$

We look at the restriction of the diagram on some of the curves in the linear system $\pi^*| - 2K_T|$. Recall $\pi^*| - 2K_T|$ is hyperelliptic, in particular, every generic curve is smooth genus 5 curve mapping 2 to 1 to a rational curve in $\mathbb{P}^2$ by $\varphi_{\pi^*(-2K_T)}$. Therefore, if we start from a curve $C' \in | - 2K_T|$, $C := \pi^{-1}(C')$ is invariant under the hyperelliptic involution i. So C is invariant under $i \circ \tau$, it descends through μ onto $\bar{S}$ as $\bar{C}$.

Because the linear system $\pi^*| - 2K_T|$ is free, as $| - 2K_T|$ is free on T, for generic curve C, it does not intersect with any of the 8 fixed points, thus the restriction of μ is unramified. By Riemann-Hurwitz formula, $g(C) = 5$, $g(\bar{C}) = 3$. Let $\eta^*(\bar{C}) = \hat{C}$, $g_a(\hat{C}) = 3$.

We have

$$\begin{tikzcd}[column sep=small] \varphi_{\pi^*(-2K_T)}|_C : & C \arrow[r, "2:1"] \arrow[d, "2:1"] \arrow[dr, "\mu"] & \mathbb{P}^1 \arrow[r, hook, "\nu_2"] & H \cap V_a \subset \mathbb{P}^5 \\ \varphi_{-2K_T}|_C : & C' & \bar{C} & \hat{C} \arrow[l, "\eta"] \end{tikzcd}$$

Lemma 6.4.2. *If C is generic in $\pi^*| - 2K_T| = \mathbb{P}^3$, $\mu : C \to \bar{C}$ is etalé, inducing a 2 to 1 map onto its image inside the Jacobian*

$$\mu^* : \operatorname{Jac} \bar{C} \to \operatorname{Jac} C$$

Furthermore, the Prym variety $P(C, \tau)$ is a double quotient of a Jacobian

$$P(C, \tau) = \mu^* \operatorname{Jac}(\bar{C})$$

Proof. Since the involution $i \circ \tau$ has 8 fixed points, a generic curve C does not pass through any of them, thus μ is etalé. To see μ^* is 2 to 1, consider a line bundle E which determines the etale double cover μ, $E^2 = \mathcal{O}_{\bar{C}}$. Then if $L_1, L_2 \in \operatorname{Jac} \bar{C}$ and $L_2 = L_1 \otimes E$, $\mu^*(L_2) = \mu^*(L_1) \otimes \mu^*(E) = \mu^*(L_1) \otimes \mathcal{O}_C$. Also see section 3 in [MUM].

Now this fits into the situation of Lemma 5.1.5, we conclude the relation on the abelian varieties $P(C, \tau) = \mu^* \operatorname{Jac}(\bar{C})$, μ^* is mapping 2 to 1 onto $P(C, \tau)$ as a subvariety in $\operatorname{Jac} C$. Notice the polarization type on $P(C,\tau)$ is $(1,2,2)$. $J(\bar{C})$ could be endowed with twice the principal polarization and the 2 to 1 map μ^* respects the polarizations.

As a corollary, since η is isomorphism for generic C, $\operatorname{Jac}\bar{C} \simeq \operatorname{Jac}\hat{C}$,

$$P(C,\tau) \simeq \mu^*\eta_* \operatorname{Jac}(\hat{C})$$

where μ^* is morphism of degree 2 and η_* is an isomorphism. $\square$

Taking out the maps on the K3 surfaces, we have

$$\begin{tikzcd} S \arrow[d, "\mu"] \arrow[dr, dashed, "\hat{\mu}"] & \\ \bar{S} & \hat{S} \arrow[l, "\eta"] \end{tikzcd}$$

the induced maps on a generic $C \in \pi^*|-2K_T|$ is following, where η is isomorphism and $\mu, \hat{\mu}$ are umramified double covers, depending on C.

$$\begin{tikzcd} C \arrow[d, "\mu"] \arrow[dr, "\hat{\mu}"] & \\ \bar{C} & \hat{C} \arrow[l, "\eta"] \end{tikzcd}$$

Induced on Jacobians, η^* is an isomorphism, and $\mu^*, \hat{\mu}^*$ are onto maps of degree 2

$$\begin{tikzcd} P(C,\tau) & \\ J(\bar{C}) \arrow[u, "\mu^*"] \arrow[r, "\eta^*"] & J(\hat{C}) \arrow[ul, "\hat{\mu}^*"'] \end{tikzcd}$$

In light of these relations, we define a rational map from the relative compactified Jacobian variety $\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$ to $\mathcal{M}_{v(C),C}(S)$, here by abuse of notation the Mukai vectors are

$$v(\hat{C}) = (0, [\hat{C}], 1 - g(\hat{C})])$$

$$v(C) = (0, [C], 1 - g(C))$$

Consider diagram

$$\begin{tikzcd} S & \tilde{S} \arrow[l, "\hat{\eta}"'] \arrow[d, "\hat{\mu}"] \\ \bar{S} \arrow[from=u, "\mu"] & \hat{S} \arrow[l, "\eta"] \end{tikzcd}$$

where $\hat{\eta}$ is the blow up of the 8 fixed points of $i \circ \tau$ on S, $\hat{\mu}$ is a double cover ramified along the (-2) curves R_i for $i = 1, \ldots 8$ in $\hat{S}$.

Theorem 6.4.3. *Take C to be a generic curve in $\pi^*|-2K_T|$, which does not pass through the 8 fixed points of $i \circ \tau$ on S. Let $\hat{C} = \eta^*(\mu(C))$. There is a rational map*

$$\phi : \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S}) \dashrightarrow \mathcal{M}_{v(C),C}(S)$$
$$F \mapsto \hat{\eta}_* \hat{\mu}^*(F)$$

defined on the open set of $\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$ parametrizing sheaves supported on integral curves which do not intersect with any of the (-2) curves R_i.

Under the inclusion $\mathcal{P}_{v(C),C}(S) \subset \mathcal{M}_{v(C),C}(S)$, ϕ is a rational dominant map

$$\phi : \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S}) \dashrightarrow \mathcal{P}_{v(C),C}(S)$$

Proof. As C varies in the open set of $\pi^*|-2K_T|$ parametrizing smooth curves, generically it does not pass through the 8 fixed points of $i \circ \tau$ on S, the map just defined

$$\hat{\mu}^* : \mathrm{Jac}(\hat{C}) \to P(C, \tau)$$

will be the definition of ϕ, as it clearly fit into a family.

Because $\hat{\mu}^*$ is surjective, it implies the map is dominant onto $\mathcal{P}_{v(C),C}(S)$.

Furthermore, $\dim \pi^*|-2K_T| = 3$, equal to the dimension of $|\hat{C}|$ on $\hat{S}$, recall $g(\hat{C}) = 3$. This means any curve in $\hat{C}$ corresponds to a curve in $\pi^*|-2K_T|$ under $\hat{\mu}$. This implies the map described above is defined on a dense open set of $\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$, namely, the open set of sheaves supported on smooth curves in $|\hat{C}|$. □

Corollary 6.4.4. *The rational dominant map $\phi : \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S}) \dashrightarrow \mathcal{P}_{v(C),C}(S)$ induces a rational involution i_ϕ on $\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$ with no fixed points on where it is defined.*

Proof. On the open set where ϕ is defined, ϕ coincides with $\hat{\mu}^*$, which by Lemma 6.4.2 is 2 to 1 everywhere, so ϕ induces a rational involution with no fixed locus. □

Theorem 6.4.5. *$\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$ is birational to a smooth Beaville-Mukai system, which is an irreducible holomorphic variety deformation equivalent to the Hilbert scheme of three points $\hat{S}^{[3]}$.*

Proof. We will first show $v(\hat{C}) = (0, [\hat{C}], -2)$ is a primitive Mukai vector. To this aim, we investigate the class of the strict transform $\hat{C}$ of $\bar{C}$ in $\mathrm{Pic}(\hat{S})$.

On degree 1 del Pezzo surface, $|-K_T|$ is a 1 dimensional family of genus 1 curve with one base point [KSC], so $\pi^*|-K_T|$ is a 1 dimensional family of genus 2 curves with 2 base points p, q. In fact, these 2 points are inside the 8 fixed points of the involution $i \circ \tau$ on S, because $\pi^*|-K_T|$ is inside the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ under $\varphi_{\pi^*(-K_T)}$, and lines in $\mathbb{P}^2$ intersect at 1 point, which is the image of both p and q under $\varphi_{\pi^*(-K_T)}$. That is to say, p and q descends onto 2 points on $\bar{S}$, as 2 of the 8 singular points of $\bar{S}$. Because p, q are ramification locus of μ, using Riemann-Roch, the image of $\pi^*|-K_T|$ under μ is a 1 dimensional family of genus 1 curves meeting in 2 points.

Now $\mu(C) = \bar{C} \sim 2\bar{C}_0$, where $\bar{C}_0 = \mu(\pi^*(-K_T))$, so $\bar{C}$ is linearly equivalent to two genus 1 curves E meeting at 2 points $\mu(p), \mu(q)$, $\bar{C} \sim 2E$, $E^2 = 2$. Then $\hat{C} = \eta^*\bar{C}$ is linearly equivalent to $\eta^*(2E) = 2F + R_1 + R_2$, where F is the strict transform of E, R_1, R_2 are the exceptional divisors of η above p, q, both are (-2)-curves. One can check the coefficients of R_1 and R_2 should be 1, hence $[\hat{C}] = [2F + R_1 + R_2]$.

If $v(\hat{C}) = (0, [\hat{C}], -2)$ is non-primitive, then $[\hat{C}] = [2C_0]$. $\hat{C}^2 = 2g(\hat{C}) - 2 = 4C_0^2$, implies $C_0^2 = 1$ which is impossible as the intersection form for K3 suffice is even.

As the Mukai vector $v(\hat{C})$ is primitive, the theory of moduli space of sheaves guarantees changing the polarization to some generic class gives a symplectic resolution of $\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$. That is a smooth irreducible holomorphic variety and a Jacobian fibration. As $g_a(\hat{C}) = 3$, dimension of moduli space is 6 and so it is birational and deformation equivalent to the Hilbert scheme of 3 points $\hat{S}^{[3]}$. □

Corollary 6.4.6. *$\hat{S}$ is in fact an elliptic K3 surface and F is the elliptic fibration. There are 8 (-2) curves R_i on $\hat{S}$, satisfying $F.R_1 = F.R_2 = 1$, $R_i.R_j = 0$, for $i \neq j$, $R_i^2 = -2$, $F^2 = 0$, and more importantly, $F.R_i = 0$ for $i = 3, \ldots, 8$.*

In fact, the two base points of $|-K_T|$ on T are mapped to the quadric cone Λ as the apex $P = [0 : 0 : 0 : 1]$ through the commutative diagram.

Proposition 6.4.7. *Let $\gamma : \tilde{\mathcal{P}} \to \mathcal{P} := \mathcal{P}_{v,C}(S)$ be any resolution of singularity, then $h^{2,0}(\tilde{\mathcal{P}}) = 1$.*

Proof. Since the Mukai vector $v(\hat{C})$ is primitive, we consider the resolution

$$\gamma : \mathcal{M}_{v(\hat{C}),H}(\hat{S}) \to \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$$

by choosing some generic polarization H. This is an isomorphism on the sheaves supported on integral curves, which contains the regular domain of $\phi : \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S}) \dashrightarrow \mathcal{P}$. Thus composing γ with ϕ, we get another rational dominant map

$$\phi \circ \gamma : \mathcal{M}_{v(\hat{C}),H}(\hat{S}) \dashrightarrow \mathcal{P}.$$

As $\mathcal{M}_{v(\hat{C}),H}(\hat{S})$ is an irreducible holomorphic symplectic variety, $h^{2,0}(\mathcal{M}_{v(\hat{C}),H}(\hat{S})) = 1$. The map $\phi \circ \gamma$ is dominant, so for a resolution

$$\pi : \tilde{\mathcal{P}} \to \mathcal{P}$$

any $\omega \in H^0(\tilde{\mathcal{P}}, \Omega^2_{\tilde{\mathcal{P}}})$ can be pulled back onto $\mathcal{M}_{v(\hat{C}),H}(\hat{S})$ via $\pi^{-1} \circ \phi \circ \gamma$, and it is defined on open set with complement having codimension greater than 1, thus can be extended to $\mathcal{M}_{v(\hat{C}),H}(\hat{S})$. Thus, $h^{2,0}(\tilde{\mathcal{P}}) \leq 1$.

Lastly it suffices to show the symplectic form on the regular locus $\mathcal{P}_{reg}$ of $\mathcal{P}$ can be extended holomorphically to $\tilde{\mathcal{P}}$. This is satisfied because $\mathcal{P}$ has symplectic singularities. □

6.5 Fundamental group of $\mathcal{P}_{v,C}$

We finally look into the fundamental group of $\mathcal{P}_{v,C}(S)$. We will use the rational dominant map $\phi : \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S}) \dashrightarrow \mathcal{P}_{v(C),C}(S)$ to further construct a smooth variety with a regular involution whose quotient is a birational model of $\mathcal{P}_{v(C),C}(S)$.

Proposition 6.5.1. *There exists a smooth variety $\tilde{\mathcal{M}}$ birational to $\mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$, such that the induced involution on $\tilde{\mathcal{M}}$ is regular, denoted by $\tilde{i}$. The birational model of $\mathcal{P}_{v,C}(S)$ is*

$$\mathcal{P}_{v,C} \overset{bir}{\simeq} \tilde{\mathcal{M}}/\tilde{i}.$$

Proof. Let us simplify notations by

$$\mathcal{M} := \mathcal{M}_{v(\hat{C}),\hat{C}}(\hat{S})$$

$$\mathcal{P} := \mathcal{P}_{v(C),C}(S)$$

We complete the proof in two steps. Firstly, from the degree 2 rational map $\phi : \mathcal{M} \dashrightarrow \mathcal{P}$, the function field $K(\mathcal{M})$ is a degree 2 extension of $K(\mathcal{P})$ and it has an involution action. Consider the integral closure of $O(\mathcal{P})$ in $K(\mathcal{M})$,

$$O(\mathcal{P})^{ic} = \{x \in K(\mathcal{M}) | x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, a_i \in O(\mathcal{P})\}$$

and take its corresponding scheme $\hat{\mathcal{M}}$.

$\hat{\mathcal{M}}$ is a normal scheme that normalizes $\mathcal{P}$

$$\hat{\phi} : \hat{\mathcal{M}} \to \mathcal{P}$$

As $O(\mathcal{P})^{ic}$ carries an involution inherited from $K(\mathcal{M})$, $\hat{\mathcal{M}}$ has a regular involution $\hat{i}$ associated to the normalization map to $\mathcal{P}$ which is compatible with the rational involution on $\mathcal{M}$.

$$\begin{tikzcd} \hat{\mathcal{M}} \arrow[r, "\hat{i}"] \arrow[d, "\hat{\phi}"] & \hat{\mathcal{M}} \arrow[dl, "\hat{\phi}"] \\ \mathcal{P} & \end{tikzcd}$$

Also, $\hat{\mathcal{M}}$ is birational to $\mathcal{M}$, as they have the same function field.

Then to ensure smoothness, we take the functorial resolution $f : \tilde{\mathcal{M}} \to \hat{\mathcal{M}}$. By functorial property of the resolution, the group action $\hat{i}$ lifts to $\tilde{i}$ on the smooth resolution $\tilde{\mathcal{M}}$.

$$\begin{tikzcd} \tilde{\mathcal{M}} \arrow[r, "\tilde{i}"] \arrow[d, "f"] & \tilde{\mathcal{M}} \arrow[d, "f"] \\ \hat{\mathcal{M}} \arrow[r, "\hat{i}"] \arrow[d, "\hat{\phi}"] & \hat{\mathcal{M}} \arrow[dl, "\hat{\phi}"] \\ \mathcal{P} & \end{tikzcd}$$

From the diagram, we get $\mathcal{P} \overset{bir}{\simeq} \tilde{\mathcal{M}}/\tilde{i}$. □

In the following where we utilize this birational model to argue the fundamental group of $\mathcal{P}$, we need an assumption that $\mathcal{P}$ only has quotient singularities. According to our previous analysis, this is satisfied except for sheaves $\mathcal{F}$ supported on non-reduced curves in $\pi^*| - 2K_T|$. The singularity of $\mathcal{M}_{v,C}(S)$ at $\mathcal{F}$ is unknown as it does not appear in OG(10), and it remains unknown what the singularity of $\mathcal{P}$ is at $\mathcal{F}$. We think it is reasonable to make this assumption considering that $\mathcal{P}$ is birational to a finite quotient of a smooth variety $\tilde{\mathcal{M}}$.

Proposition 6.5.2. *Under the assumption that $\mathcal{P}$ only has quotient singularities, the relative Prym variety $\mathcal{P}_{v,C}(S)$ has the same fundamental group as a quotient space of a simply-connected space by a group of order at most 2.*

Proof. Consider diagram

$$\begin{tikzcd} \tilde{\mathcal{P}} \arrow[r, "\overset{bir}{\simeq}"] \arrow[d, "\gamma"] & \mathcal{M}' \arrow[d, "\xi"] \\ \mathcal{P} \arrow[r, "\overset{bir}{\simeq}"] & \tilde{\mathcal{M}}/\tilde{i} \end{tikzcd}$$

where γ, ξ are resolutions of singularities. Since $\mathcal{P}$ and $\tilde{\mathcal{M}}/\tilde{i}$ have quotient singularities, the resolutions do not change the fundamental group. As $\tilde{\mathcal{P}}$ and $\mathcal{M}'$ are smooth and birational, by the Weak Factorization Theorem,

$$\pi_1(\tilde{\mathcal{P}}) = \pi_1(\mathcal{M}'),$$

thus

$$\pi_1(\mathcal{P}) = \pi_1(\tilde{\mathcal{M}}/\tilde{i})$$

Notice $\tilde{\mathcal{M}}$ is smooth and birational to $\mathcal{M}$, and thus birational to the smooth holomorphic symplectic variety $\mathcal{M}_{v(\hat{C}),H}(\hat{S})$, so $\pi_1(\tilde{\mathcal{M}}) = \pi_1(\hat{S}^{[3]}) = 0$.

At this point, we cannot say more about $\pi_1(\mathcal{P})$. But if the fixed locus of $\tilde{i}$ on $\tilde{\mathcal{M}}$ is non-empty, then we can conclude $\pi_1(\mathcal{P}) = 0$, because the natural homomorphism $\pi_1(\tilde{\mathcal{M}}) \to \pi_1(\tilde{\mathcal{M}}/\tilde{i})$ is surjective, see Lemma 1.2 in [F]. Otherwise, $\pi_1(\mathcal{P}) = \mathbb{Z}_2$.

□

CHAPTER 7

Duality between two relative Prym varieties

One motivation to study the relative Prym associated to degree 1 del Pezzo is the dual fibrations Menet constructed. Menet showed that for two degree two del Pezzo surfaces as double cover of $\mathbb{P}^2$ branched along a quartic, when the two quartics are bitangent, the two corresponding relative Prym varieties have dual Lagrangian fibrations. We wonder whether there exists other pairs of low dimensional relative Prym varieties from del Pezzo surfaces such that generic fibers are dual abelian varieties.

The dual of an abelian variety with principal polarization is isomorphic to itself. In general, the dual of an abelian variety A is defined as

$$\mathrm{Pic}^0(A)$$

The dual of an n dimensional abelian variety of type $(1, \dots, 1, 2, \dots, 2)$ with g 2's is of type $(1, \dots, 1, 2, \dots, 2)$ with $(n - g)$ 2's.

Denote the relative Prym variety associated to a degree 3 del Pezzo surface in Matteini's example by $\mathcal{P}_3$ and the new relative Prym associated to degree 1 del Pezzo surface and linear system $|\pi^*(-2K_T)|$ by $\mathcal{P}_1$, see previous chapter for details. Interestingly, $\mathcal{P}_3$ and $\mathcal{P}_1$ have the same dimension and the polarization type of generic fibers are dual. We conjecture that $\mathcal{P}_3$ and $\mathcal{P}_1$ are dual fibrations.

7.1 The Prym map

Let

$$f : C \to D$$

be a double cover of a genus g curve D ramified along r points, r is even. Denote by $\mathcal{R}_{g,r}$ the set of such pairs.

The Prym variety of this map $Prym(C/D)$ is an abelian variety of dimension $d = g - 1 + r/2$, polarization type $\delta = (1, ..., 1, 2, ..., 2)$, where the number of 2 among the d entries is g.

Denote the moduli space of double covers of a genus g curve ramified along r points by $\mathcal{R}_{g,r}$ and moduli space of dimension d abelian variety with polarization δ by A_d^δ. The Prym map is defined as

$$Prym : \mathcal{R}_{g,r} \to A_d^\delta.$$

mapping a double cover to its Prym variety.

We consider

$$\mathcal{R}_{2,4}$$

containing double covers from a genus 5 curve to a genus 2 curve with 4 branch points. $\mathcal{R}_{g,r} = 3g-3+r, g > 1$, so $\dim \mathcal{R}_{2,4} = 7$. Under the Prym map, it is mapped to $A_3^{(1,2,2)}$ and the dual abelian variety lies in $A_3^{(1,1,2)}$ which are both 6 dimensional moduli spaces.

By [MN], the Prym map

$$Prym : \mathcal{R}_{1,6} \to A_3^{(1,1,2)}$$

is generically injective, so the dual abelian variety can be uniquely realized as the Prym variety of an element in $\mathcal{R}_{1,6}$, double covers of elliptic curves branched at 6 points.

In summary, we have the following diagram

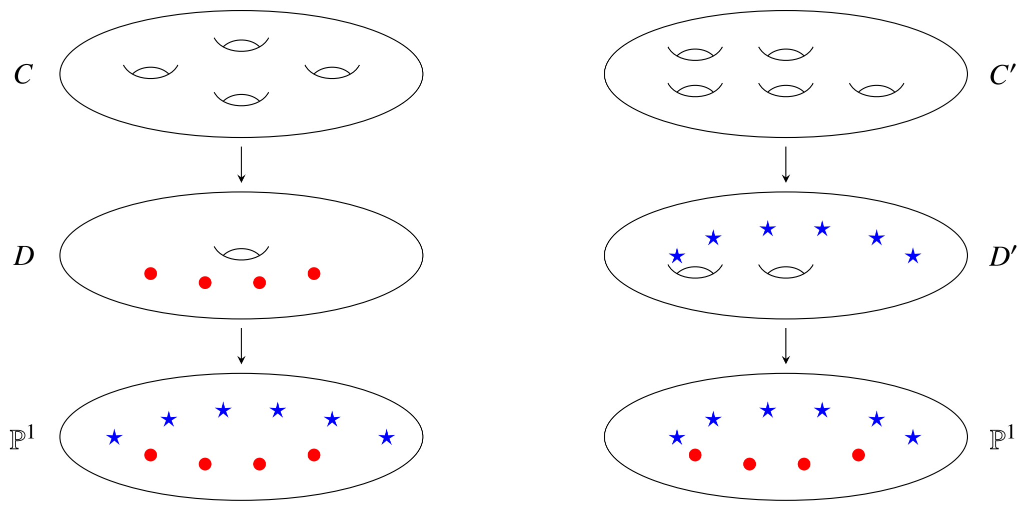
$$\begin{tikzcd}
\mathcal{R}_{2,4} \arrow[r] \arrow[d, "1-dim"] & \mathcal{R}_{1,6} \arrow[d, "inj"] \\
A_3^{(1,2,2)} \arrow[r, "dual"] & A_3^{(1,1,2)}
\end{tikzcd}$$

7.2 Pantazis's construction

In fact, the reverse direction $\mathcal{R}_{1,6} \to \mathcal{R}_{2,4}$ could be realized by Pantazis's construction, see [P].

Let $C \to D$ be a double cover from a genus 4 curve to a genus 1 curve ramified at 6 points, as genus 1 curves are all hyperelliptic, D maps to $\mathbb{P}^1$ with 4 branch points.

Switch the two branch loci on $\mathbb{P}^1$, and let $D' \to \mathbb{P}^1$ be a double cover ramified at 6 points, $C' \to D'$ is a double cover from genus 5 curve to genus 2 curve ramified along 4 points.



The Lagrangian fibrations of the Prym varieties $\mathcal{P}_3$ is a 3 dimensional family of Prym varieties of type $A_3^{(1,1,2)}$. Under this correspondence, we should have a 3 dimensional family of Prym varieties of type $A_3^{(1,1,2)}$ on the other side.

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{P}_1} \arrow[d] & \mathcal{M}_{\mathcal{P}_3} \arrow[d] \\
\mathcal{R}_{2,4} \arrow[r] \arrow[d, "1-dim"] & \mathcal{R}_{1,6} \arrow[d, "inj"] \\
A_3^{(1,2,2)} \arrow[r, "dual"] & A_3^{(1,1,2)}
\end{tikzcd}$$

The question is whether the family comes from the Prym fibration $\mathcal{P}_1$ associated to some degree 1 del Pezzo surface. Recall to construct $\mathcal{P}_1$, we take $\pi : S' \to T'$ a branched double cover of degee 1 del Pezzo suface and D' a generic element in $|-2K_T|$. Restricting $S' \to T' \to \Lambda$ on curves, $C' \to D' \to \mathbb{P}^1$ is a tower of double covers from genus 5 curve over genus 2 curve ramified at 4 points, and to a rational curve branched along 6 points.

Question 7.2.1. *Is there any pair of degree 1 del Pezzo surface T_1 and degree 3 del Pezzo surface T_3 such that the relative Prym variety $\mathcal{P}_1$ associated to $\pi^*(-2K_{T_1})$, where $\pi : S \to T_1$ is a double cover by a K3 surface S, and $\mathcal{P}_3$ associated to $\pi^*(-K_{T_3})$, where $\pi : S' \to T_3$ is a double cover by a K3 surface S', satisfy the dual relation on their Lagrangian fibrations $\pi_1 : \mathcal{P}_3 \to \mathbb{P}^3$ and $\pi_2 : \mathcal{P}_1 \to \mathbb{P}^3$, i.e. for generic $x \in \mathbb{P}^3$, $\pi_1^{-1}(x)$ and $\pi_2^{-1}(x)$ are dual abelian varieties?*

Furthermore, notice if the family of double covers of curves in $\mathcal{R}_{2,4}$ are from the Prym fibration of $\mathcal{P}_1$, there is something special on $C' \to D' \to \mathbb{P}^1$, namely, the 4 branched points of $C' \to D'$ mapped to $\mathbb{P}^1$ are 2 points with multiplicity 2. If we apply Pantazis's construction on such double cover, we get a double cover of singular curves, which resembles the double cover in the singular locus of of $\mathcal{P}_3$. Thus another question arises,

Question 7.2.2. *Is $\mathcal{P}_1$ dual to a degeneration of $\mathcal{P}_3$?*

We have not fully understood the duality and would leave it as a future direction.

CHAPTER 8

Degeneration between two Lagrangian fibrations

In this section, we present an extension of the work by Ron Donagi, Lawrence Ein, Robert Lazarsfeld in [DEL] where they proposed a way to relate the classical Hitchin system and the Beauville-Mukai system.

Theorem 8.0.1. *If the K3 surface S has a hyperplane section C which is a curve of genus g, then based on a deformation of S to the normal cone of C in S, the Beauville-Mukai system*

$$\mathcal{M} = \mathcal{M}^k_{|nC|}(S) = \mathcal{M}_{v,C}(S), v = (0, [nC], k + 1 - g)$$

$$\mathcal{M} \to \bar{B} = \mathbb{P}^{\tilde{g}}$$

where $\tilde{g} = n^2(g-1)+1$, has a non-linear deformation to a natural compactification of the $\mathrm{GL}(n)$*-Hitchin system $\mathcal{H}$ consisting of stable pairs (E, ϕ), where E is a rank n degree d stable vector bundle on C, d and k satisfy $k = d + (n^2 - n)(g-1)$*

$$\mathcal{H} \to \mathbb{C}^{\tilde{g}}$$

This degeneration is given by a support map of schemes over $\mathbb{P}^1$

$$\bar{\mathcal{W}} \to \bar{\mathcal{B}}$$

which satisfies for $t \neq 0 \in \mathbb{P}^1$, the fiber

$$[\bar{\mathcal{W}}_t \to \bar{\mathcal{B}}_t] \cong [\mathcal{M} \to \mathbb{P}^{\tilde{g}}]$$

and

$$[\bar{\mathcal{W}}_0 \to \bar{\mathcal{B}}_0] \cong [\bar{\mathcal{H}} \to \mathbb{P}^{\tilde{g}}].$$

We describe a degeneration between the $\mathrm{Sp}(2n)$-Hitchin system $\mathcal{H}$ on a curve C of genus g and a relative Prym variety $\mathcal{P}$ associated to a double cover of K3 surface S and a del Pezzo surface T. In the end, we have a degeneration from a natural compactification of a non-compact Lagrangian fibration to a compact Lagrangian fibration, both fibers being non-Jacobian.

First of all, for such degeneration to exist, we need to require that the curve C and the K3 surface S be geometrically related, that is, S is a double cover of some del Pezzo surface T with degree d, the branch locus Δ_T of π is in the linear system $|-2K_T|$ and $C = \Delta_S$ is the preimage of Δ_T

$$\pi : S \xrightarrow{2:1} T, \Delta_T$$

$$\Delta_S = C \mapsto \Delta_T$$

Furthermore, one can show that when this happens, the degree of the del Pezzo surface satisfies $d = g - 1$, where g is the genus of the curve C. We will assume this setting through out the whole section.

Theorem 8.0.2. *Let $\pi : S \to T, \Delta_T$ be a double cover from a K3 surface S to a del Pezzo surface T of degree d ramified over a curve $\Delta_T \subset T$. Let $C = \pi^{-1}(\Delta_T)$, then $g(C) = g = d + 1$.*

Denote by $\mathcal{H}$ the $\mathrm{Sp}(2n)$*-Hitchin system of stable pairs on C consisting of a rank $2n$, degree d vector bundle and a Higgs field ϕ, the Hitchin map is a Lagragian fibration*

$$\mathcal{H} \to \mathbb{C}^{\tilde{g}}$$

where $\tilde{g} = n(2n + 1)g$.

And denote by $\mathcal{P}$ the relative Prym variety $\mathcal{P}_{v,C}(S)$ constructed from $\pi : S \to T, \Delta_T$ and a smooth curve in $\pi^*|-2nK_T| \subset |2nC|$, using $v = (0, [2nC], k + 1 - g(2nC))$ and $k = d + ((2n)^2 - 2n)(g - 1)$. The support map is a Lagrangian fibration

$$\mathcal{P} \to \mathbb{P}^{\tilde{g}}$$

Then there is a degeneration from $\mathcal{P}$ to a natural compactification of $\mathcal{H}$ given by a support map of schemes over $\mathbb{P}^1$

$$\bar{\mathcal{W}} \to \bar{\mathcal{B}}$$

which satisfies for $t \neq 0 \in \mathbb{P}^1$, the fiber

$$[\bar{\mathcal{W}}_t \to \bar{\mathcal{B}}_t] \cong [\mathcal{P} \to \mathbb{P}^{\tilde{g}}]$$

and

$$[\bar{\mathcal{W}}_0 \to \bar{\mathcal{B}}_0] \cong [\bar{\mathcal{H}} \to \mathbb{P}^{\tilde{g}}].$$

For the proof, we will first find a map defining a relative double cover

$$\mathcal{S} \xrightarrow{\pi} \mathcal{T} \xrightarrow{f} \mathbb{P}^1$$

such that fiber over $x = 0 \in \mathbb{P}^1$ gives

$$\overline{K_C} \xrightarrow{\pi_0} \overline{K_{\Delta_T}}^2$$
$$f\mathrm{d}z \mapsto f^2\mathrm{d}z^2$$

where $\overline{K_C}$ denotes the one point compactification of K_C, which is $\mathbb{P}(K_C \oplus \mathcal{O}_C)$ with the infinity section blown down to a point, and fiber over $x \neq 0 \in \mathbb{P}^1$ gives maps isomorphic to

$$S \xrightarrow{\pi_1} T$$

Then we will display the degeneration between the two fibrations by showing:

1. π_0 yields the spectral data of the Sp$(2n)$-Hitchin system on C $\mathcal{H} \to B$;
2. π_1 and linear system $\pi^*|-2nK_T|$ produces the relative Prym fibration $\mathcal{P} \to \bar{B}$;
3. under a natural compactification, the base spaces of the two fibration fit into a $\mathbb{P}^N$-bundle $\bar{\mathcal{B}} \to \mathbb{P}^1$;
4. deformation π eventually leads to the deformation $\bar{\mathcal{W}} \to \bar{\mathcal{B}}$ parametrized by $\mathbb{P}^1$, with central fiber being $\bar{\mathcal{H}} \to \bar{B}$ and other fibers isomorphic to $\mathcal{P} \to \bar{B}$.

8.1 Degeneration of double cover

We show that the $K3$ surface containing the curve C can be degenerated to the one-point compactification $\overline{K_C}$ of the canonical bundle of C, namely, the cone over the canonical embedding of the curve [DEL].

Consider blowing up $C \times \{0\}$ inside $S \times \mathbb{P}^1$

$$\epsilon_1 : \widehat{S \times \mathbb{P}^1} \to S \times \mathbb{P}^1$$

The exceptional divisor of $C \times \{0\}$ under ϵ_1 is $\mathbb{P}(N_{C\times\{0\}\subset S\times\mathbb{P}^1})$ the projectivized space of the normal bundle of $C \times \{0\}$ in $S \times \mathbb{P}^1$.

As $C \subset S \subset S \times \mathbb{P}^1$, and $S \times \mathbb{P}^1$ is a trivial fibration

$$N_{C\times\{0\}\subset S\times\mathbb{P}^1} \simeq N_{C\subset S} \oplus N_{S\times\{0\}\subset S\times\mathbb{P}^1}|_C$$

Also $N_{C\subset S} \simeq O_S(C)|_C \simeq K_C$ and $N_{S\times\{0\}\subset S\times\mathbb{P}^1} \simeq O_S$, thus

$$\mathbb{P}(N_{C\times\{0\}\subset S\times\mathbb{P}^1}) \simeq \mathbb{P}(K_C \oplus O_C)$$

Composed with projection pr_2

$$pr_2 \circ \epsilon_1 : \widehat{S \times \mathbb{P}^1} \to S \times \mathbb{P}^1 \to \mathbb{P}^1$$

its fiber over $0 \in \mathbb{P}^1$ is the union of the strict transform of S under ϵ_1 and the exceptional divisor $\mathbb{P}(K_C \oplus O_C)$, which intersect at the section corresponding to K_C as a sub-bundle of $K_C \oplus O_C$ inside the projectivization $\mathbb{P}(K_C \oplus O_C)$, because this section represents the direction in S normal to C. Let's call it the infinity section. The zero section is the intersection of the strict transform of $C \times \mathbb{P}^1$ with $\mathbb{P}(K_C \oplus O_C)$.

Consider the strict transform of $S \times \{0\}$ and still call it S, as $N_S^* = O_S(C)$ is ample, we can blow-down the strict transform of S inside the fiber above 0. Due to a result by M. Artin, this only gives an algebraic space, but in this case the resulting space is indeed an algebraic variety as described in [DEL]. Denote the blow-down by ϵ_2

$$\epsilon_2 : \widehat{S \times \mathbb{P}^1} \to \mathcal{S}$$

$\mathcal{S}$ is the family we look for, because under ϵ_2 the infinity section of $\mathbb{P}(K_C \oplus O_C)$ together with the strict transform of S is blown down to a point. We know the resulting space is the cone over the canonical embedding of C, i.e. the one-point compactification of K_C, denoted by $\overline{K_C}$.

Hence, consider the induced map

$$f = pr_2 \epsilon_1 \epsilon_2^{-1} : \mathcal{S} \to \mathbb{P}^1$$

f is a regular map as the exceptional divisor of ϵ_2 is contained in the 0-fiber of $\widehat{S \times \mathbb{P}^1}$. Furthermore, f satisfies $f^{-1}(0) = \overline{K_C}$ and $f^{-1}(t) \simeq S, t \neq 0$. Thus f is a degeneration between S and $\overline{K_C}$.

Along this family $\mathcal{S} \to \mathbb{P}^1$, the involution on S induced by the double cover $\pi : S \to T$, call σ, degenerates to an involution on the central fiber $\overline{K_C}$. Firstly, the induced involution along ϵ_1 under some appropriate coordinate can be represented by multiplication by -1 on the K_C factor because C is the fixed curve of the involution. Then pushed onto $\overline{K_C}$, it becomes

$$\begin{aligned} \sigma : \overline{K_C} &\to \overline{K_C} \\ f\mathrm{dz} &\mapsto -f\mathrm{dz} \end{aligned}$$

As it is invariant under the squaring operation, the quotient space of $\overline{K_C}$ by this action is

$$\pi_0 : \overline{K_C} \xrightarrow{2:1} \overline{K_C}^2$$

So $\mathbb{P}^1$ simultaneously parametrizes a family of quotient spaces, we denote by $\mathcal{T} \to \mathbb{P}^1$ because fibers above non-zero are isomorphic to the del Pezzo surface T, and the central fiber is $\overline{K_C}^2$. We get the commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[d, "\pi"] \arrow[dr, "f"] & \\ \mathcal{T} \arrow[r, "f'"] & \mathbb{P}^1 \end{tikzcd}$$

A remark is that this can be verified from another perspective, by applying the same construction downstairs on the del Pezzo surface T.

Consider blowing up $\Delta_T \times \{0\}$ in $T \times \mathbb{P}^1$,

$$\epsilon_1' : \widehat{T \times \mathbb{P}^1} \to T \times \mathbb{P}^1$$

As $N_{\Delta_T \subset T} = \mathcal{O}(\Delta_T)|_{\Delta_T} = -2K_T|_{\Delta_T}$ and $K_{\Delta_T} = \mathcal{O}_T(\Delta_T) + K_T = -K_T$ we have the exceptional divisor

$$\mathbb{P}(N_{\Delta_T \times \{0\} \subset T \times \mathbb{P}^1}) = \mathbb{P}(K^2_{\Delta_T} \oplus \mathcal{O}_T)$$

Then blow-down the strict transform of $T \times \{0\}$ inside its union with the exceptional divisor $\mathbb{P}(K^2_{\Delta_T} \oplus \mathcal{O}_T)$,

$$\epsilon'_2 : \widehat{T \times \mathbb{P}^1} \to \mathcal{T}$$

the composition

$$f' = pr_2 \epsilon'_1 \epsilon'^{-1}_2 : \mathcal{T} \to \mathbb{P}^1$$

satisfies $f'^{-1}(0) = \overline{K^2_{\Delta_T}}$ and $f'^{-1}(t) \simeq T, t \neq 0$.

8.2 Spectral data

This section helps to better see the relation of the double cover of curves in the two systems.

Recall the spectral data of $\mathrm{Sp}(2n, \mathbb{C})$-Hitchin system, see diagram (8.1). The Hitchin map $h : \mathcal{H} \to B$ is a fibration over $B = \bigoplus_{i=1}^n H^0(C, K_C^{2i})$.

For a generic point $p \in B$, it corresponds to a spectral curve D_p in the total space of K_C with equation

$$x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n} = 0$$

it is mapping 2n to one to C, which is the zero section of K_C. As the equation is invariant under an involution $\sigma(x) = -x$ on K_C, D_p maps to a curve D_p/σ in the total space of K_C^2 which is an n to one cover to C, the zero section of K_C^2 .

The fiber of the Hitchin map over p, $h^{-1}(p)$ is the Prym variety of the map $D_p \to D_p/\sigma$, which is the set of line bundles in $\mathrm{Jac}(D_p)$ satisfying $(-\sigma^*)L = L$, where $-L = L^*$.

$$\begin{tikzcd}
D_p \arrow[rr, hook] \arrow[dr, "2n:1"] \arrow[dd, "2:1"'] & & K_C \arrow[dd, "2:1"] \\
& C \arrow[ur, hook] \arrow[dd] & \\
D_p/\sigma \arrow[rr, hook] \arrow[dr, "n:1"] & & K_C^2 \\
& C \arrow[ur, hook] &
\end{tikzcd} \tag{8.1}$$

$$\begin{tikzcd}
\mathcal{H} \arrow[d, "h"] & & \mathrm{Prym}(D_p/(D_p/\sigma)) \arrow[d, mapsto, "h"] \\
B & & p
\end{tikzcd} \tag{8.2}$$

On the compact side, the K3 surface S is a double cover of a del Pezzo surface T branched over a smooth curve $\Delta_T \subset T$, $\Delta_S = \pi^{-1}(\Delta_T)$.

In the construction of the relative Prym variety, we take the sub-linear system of curves in $|2n\Delta_S|$ invariant under the involution associated to the double cover, which is in one-one correspondence with the linear system $|-2nK_T|$.

We take a smooth curve $\Gamma' \in |-2nK_T|$ and $\Gamma = \pi^{-1}(\Gamma')$. As

$$\pi^*(\Delta_T) \sim \pi^*(-2K_T) \sim 2\Delta_S$$

$$\Gamma = \pi^*(\Gamma') \sim \pi^*(-2nK_T) \sim 2n\Delta_S$$

Following the construction of the relative Prym of this double cover, we end up with a Lagrangian fibration of a compact holomorphic symplectic variety, of which the base space is the linear system of curves $|-2nK_T|$ and a generic fiber is the Prym variety of the double cover of smooth irreducible curves.

$$
\begin{tikzcd}
\Gamma \in |2n\Delta_S| \arrow[rr, hook] \arrow[dd, "2:1"] & & S \arrow[dd, "2:1"] \\
& \Delta_S \arrow[ur, hook] \arrow[dd] & \\
\Gamma' \in |-2nK_T| \arrow[rr, hook, "1:1"] & & T \\
& \Delta_T \in |-2K_T| \arrow[ur, hook] &
\end{tikzcd}
\tag{8.3}
$$

$$
\begin{tikzcd}
\mathcal{P} \arrow[d] & Prym(\Gamma/\Gamma') \arrow[d, mapsto] \\
B & \{\Gamma'\}
\end{tikzcd}
\tag{8.4}
$$

Diagram (8.1)-(8.4) well illustrate the correspondence between the double cover of curves in the two settings. The family of double cover of surfaces in the last section

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "\pi"] \arrow[dr, "f"] & \\
\mathcal{T} \arrow[r, "f'"] & \mathbb{P}^1
\end{tikzcd}
$$

is also a degeneration on the curve level.

8.3 Dimension of the fiber

We show the fiber dimension of the two fibrations are the same.

We have seen for Sp($2n$)-Hitchin system, the base space is

$$B = \bigoplus_{i=1}^{n} H^0(C, K_C^{2i})$$

To compute its dimension, we use the Riemann-Roch formula,

$$h^0(C, K_C^i) = \dim H^0(C, K_C^i) = \chi(K_C^i) = i\deg(K_C) + 1 - g$$

Taking the sum for $i = 2, 4, \ldots, 2n$, we get

$$\dim B = n(2n+2)(g-1) + n(1-g) = n(2n+1)(g-1)$$

so the base space of the Sp($2n$)-Hitchin system is $\mathbb{C}^{n(2n+1)(g-1)}$ and the fibers are complex tori with the same dimension.

For the relative Prym associated to the double cover of del Pezzo surface, using Serré duality and Kodaira vanishing theorem,

$$\begin{aligned}
\dim|-2nK_T| &= h^0(T,-2nK_T)-1=\chi(T,-2nK_T)-1\\
&=\frac{(-2nK_T)^2-(-2nK_T).K_T}{2}+\chi(T,\mathcal{O}_T)-1\\
&=n(2n+1)d
\end{aligned}$$

here we used $\chi(T)=\chi(\mathbb{P}^2)=1$.

As Γ' varies in the linear system, we take the Prym variety of the map $\pi:\Gamma\to\Gamma'$, which is a complex torus with dimension $g(\Gamma)-g(\Gamma')$. We have

$$\begin{aligned}
2g(\Gamma)-2 &= \Gamma.\Gamma+\Gamma.K_S\\
&=2(\Gamma')^2=2(2n)^2K_T^2\\
2g(\Gamma')-2 &= \Gamma'.\Gamma'+\Gamma'.K_T\\
&=(2n)^2K_T^2-(2n)K_T^2=2n(2n-1)K_T^2
\end{aligned}$$

so

$$\begin{aligned}
g(\Gamma) &= (2n)^2d+1\\
g(\Gamma') &= n(2n-1)d+1\\
\dim \mathrm{Prym}(\Gamma/\Gamma') &= n(2n+1)d
\end{aligned}$$

One can also check Riemann-Hurwitz on the double cover of curves $\pi:\Gamma\to\Gamma'$

$$2(2-2g(\Gamma'))-\Gamma'.\Delta_T=2-2g(\Gamma)$$

Thus the relative Prym is a Lagrangian fibration over $|\Gamma'|=|-2nK_T|=\mathbb{P}^{n(2n+1)d}$ and fibers are complex tori of equal dimension.

Lastly, the relation $d = g - 1$ is forced to hold by applying the Riemman-Roch formula on Δ_S in S

$$2g - 2 = 2g(\Delta_S) - 2 = \Delta_S.(\Delta_S + K_S)$$
$$= (\frac{1}{2}\pi^*(\Delta_T))^2 = 2(K_T)^2 = 2d$$

so denote

$$\tilde{g} = n(2n + 1)d = n(2n + 1)(g - 1)$$

we achieved that Sp($2n$)-Hitchin system is a $\mathbb{P}^{\tilde{g}}$-fibration $\mathcal{H} \to B = \mathbb{C}^{\tilde{g}}$ and the relative Prym associated to degree d del Pezzo is a $\mathbb{P}^{\tilde{g}}$-fibration $\mathcal{P} \to \bar{B} = \mathbb{P}^{\tilde{g}}$.

8.4 Degeneration between two Lagrangian fibrations

Previously we showed the double cover $\pi : S \to T$ can be degenerated to $\overline{K_{\Delta_S}} \to \overline{K^2_{\Delta_T}}$ (or $\overline{K_C} \to \overline{K^2_C}$, as $\Delta_S = C \in S$ and Δ_T is the branch locus). Moreover, when the spectral curve in the Hitchin system does not pass through infinity, the cover of curves in the construction of the relative Prym degenerates to the spectral data in the Hitchin system. The two resulting Lagrangian fibration have same fiber dimension, but the base spaces are off by a compactification. In this section, we explain a natural compactification on the Hitchin system which completes our proof of the degeneration between the Lagrangian fibrations.

Let us view $\overline{K_C}$ as the cone over the canonical embedding of C. In the Sp($2n$)-Hitchin sytem, the spectral curves $D_p \subset \overline{K_C}$ belong to the linear system $|2nC|$ on $\overline{K_C}$. The spectral curves not passing through the apex of $\overline{K_C}$ are parametrized by the Hitchin base $\mathbb{C}^{n(2n+1)(g-1)}$. If we allow spectral curves to pass through the apex, Hitchin base space becomes

$$\overline{B} = \overline{\mathbb{C}^{n(2n+1)(g-1)}} = \mathbb{P}^{n(2n+1)(g-1)}$$

and the Sp($2n$)-Hitchin moduli space can be extended as

$$\overline{\mathcal{H}} = \overline{\mathcal{M}^k_{|2nC|}(\overline{K_C})}^{\sigma}$$

which is the moduli space of semi-stable pure sheaves of dimension one on $\overline{K_C}$ invariant under the involution $-\sigma^*$, where -1 is the relative version of taking the dual sheaf as mentioned in 5.2. They are sheaves supported on curves in $|2nC|$ having the same Hilbert polynomial as a degree k line bundle on a smooth curve in $|2nC|$.

Here $k = d + ((2n)^2 - (2n))(g-1)$, as a semi-stable rank $2n$ vector bundle of degree d supported on C can be viewed as a degree $d + ((2n)^2 - (2n))(g-1)$ line bundle supported on an element in $|2nC|$, see [DEL, SIM].

Hence we have a naturally compactified $\mathrm{Sp}(2n)$-Hitchin system

$$\bar{h} : \overline{\mathcal{H}} \to \overline{B}$$

Using the degeneration f we constructed before

$$\mathcal{S} \xrightarrow{\pi} \mathcal{T} \xrightarrow{f} \mathbb{P}^1$$

let

$$\bar{\mathcal{B}} \to \mathbb{P}^1$$

be the $\mathbb{P}^{\tilde{g}}$-bundle whose fiber over $x \in \mathbb{P}^1$ is

$$|\pi^* \mathcal{O}_{\mathcal{T}_x}(nC)|, \mathcal{T}_x = f^{-1}(x)$$

and let

$$\bar{\mathcal{W}} \to \mathbb{P}^1$$

be the relative moduli space whose fiber over $x \in \mathbb{P}^1$ is the moduli space constructed on $\pi : \mathcal{S}_x \to \mathcal{T}_x$, the fibers of $\mathcal{S} \xrightarrow{\pi} \mathcal{T} \xrightarrow{f} \mathbb{P}^1$ over x, namely

$$\mathcal{M}^k_{|2nC|}(\mathcal{S}_x)^{\sigma}$$

the moduli space of semi-stable pure sheaves of dimension one on $\mathcal{S}_x \cong S$ invariant under the involution $-\sigma^*$, where -1 is the relative version of taking the dual sheaf as mentioned in 5.2. They are supported on a curve in $|2nC|$ having the same Hilbert polynomial as a degree k line bundle on a smooth curve in $|2nC|$. By definition of the relative Prym variety, this is just the relative Prym variety $\mathcal{P}_{v,C}(S)$ constructed from $\pi : S \to T, \Delta_T$ and a smooth curve in $\pi^*|-2nK_T| \subset |2nC|$, using $v = (0, [2nC], k + 1 - g(2nC))$ and $k = d + ((2n)^2 - 2n)(g-1)$.

The support map globalizes to

$$\bar{\mathcal{W}} \to \bar{\mathcal{B}} \to \mathbb{P}^1$$

The arguments before shows that for $x = 0 \in \mathbb{P}^1$ the fiber is

$$\overline{\mathcal{H}} \to \bar{B}$$

and for $x \neq 0 \in \mathbb{P}^1$, the fiber is isomorphic to

$$\mathcal{P} \to \bar{B}.$$

CHAPTER 9

Miscellaneous

In data science, various algorithms are developed to solve the problem: given a dataset, what is the best model behind it? Depending on the specific nature of the data, solutions are given in a seemingly unrelated form. But under the geometric point of view, the problem can be unified into: given a set of points, what is the underlying manifold that gives the smallest error?

Information geometry is a branch of study which makes such statement rigorous, and the problem becomes: given a dataset from some statistical model, on the manifold of all statistical models, which point has the shortest distance to the empirical distribution?[IG]

The manifold M of statistical models is

$$\{p : X \to \mathbb{R} \mid p(x) > 0, \int_X p(x) = 1\}$$

usually restricted to some parametric subspace. Then one can choose a type of distance for specific needs. For example, the distance induced from the Kullback-Leibler divergence, for $p, q \in M$

$$D_{KL}(p, q) = \int p(x) \log \frac{p(x)}{q(x)} dx.$$

The following methods all fall into this category. [ESL]

Regression models

1. Ordinary least square regression

2. Linear regression

3. Logistic regression

Regularization

1. Ridge regression

2. Least Absolute Shrinkage and Selection Operator

Decision tree

1. Classification and Regression Tree

2. Random forest

3. Gradient Boosting Tree

The complexity in the geometry could increase tremendously after ensembles and alternations on the model, and it leaves an interesting question to give theoretical proof to the existence and convergence of the solution.

However, in real-life problems, we noticed high complexity models may not surpass simple but robust models, because of the noise and restrictions on the data collected and also the fact that the assumptions one has to make are not satisfied in general at all.

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